Hankel matrices of indeterminate moment problems

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Spectral theory of Hankel operators and related topics
King's College London, November 2–3, 2017

Based on joint work with
Ryszard Szwarc (Wrocław)

Overview

1. Positive definite Hankel matrices $\mathcal{H}$ and their sections $\mathcal{H}_N$

2. Hamburger moment problems: Determinacy versus indeterminacy

3. Behaviour of the smallest eigenvalue of $\mathcal{H}_N$

4. A digression about the Hilbert matrix

5. The reproducing kernel of an indeterminate moment problem and the infinite matrix $\mathcal{A}$

6. Is $\mathcal{A}$ an inverse of $\mathcal{H}$?

Infinite Hankel matrices

Given a sequence $(s_n)_{n\geq 0}$ of real numbers, we consider the infinite Hankel matrix $\mathcal{H} = \{s_{j+k}\}_{j,k=0}^{\infty}$

$$\mathcal{H} = \begin{pmatrix} s_0 & s_1 & s_2 & \cdots \\ s_1 & s_2 & s_3 & \cdots \\ s_2 & s_3 & s_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Hamburger proved around 1920 that $\mathcal{H}$ is positive definite in the sense that

$$(\mathcal{H}v, v) > 0, \quad v = (v_0, v_1, \ldots, v_n, 0, 0, \ldots), v \neq 0,$$

if and only if

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0 \tag{1}$$

for a positive measure μ on $\mathbb{R}$ with infinite support.

The finite truncations

An equivalent formulation is that all the finite sections

$$\mathcal{H}_N = \{s_{j+k}\}, 0 \le j, k \le N$$

are positive definite matrices or that

$$D_N = \det \mathcal{H}_N > 0, \quad N = 0, 1, \ldots .$$

The moment sequence (s_n) can be determinate or indeterminate in the sense that the moment equation can have exactly one solution μ or several solutions μ.
Hamburger gave various characterizations of determinacy, I will come back to some of them. He missed however the following:

A theorem of B, Chen and Ismail

Let λ_N be the smallest eigenvalue of of the positive definite matrix $\mathcal{H}_N$,

$$\lambda_N = \min\{(\mathcal{H}_N v, v) \mid v \in \mathbb{R}^{N+1}, ||v|| = 1\} > 0.$$

Therefore $\lambda_N \geq \lambda_{N+1}$, hence

$$\lambda_\infty := \lim_{N\to\infty} \lambda_N \text{ exists, and } \quad \lambda_\infty \geq 0.$$

The number λ_∞ characterizes determinacy:

Theorem (B-C-I, Math. Scand. 2002)

$\lambda_\infty = 0$ *if and only if* (s_n) *is determinate.*

Photo of Hans Ludwig Hamburger (1889-1956)

Interesting biography of Hamburger in MacTutor History of Mathematics archive at St. Andrews

The orthonormal polynomials associated with (s_n) and μ

Gram-Schmidt procedure to $1, x, x^2, \ldots$ in $L^2(\mu)$ leads to an orthonormal sequence $P_n(x), n = 0, 1, \ldots$, i.e.

$$\int P_n(x)P_m(x)\, d\mu(x) = \delta_{nm}. \quad (*)$$

Notice: (P_n) is independent of the choice of μ in the indeterminate case.

The assumption $P_n(x)$ polynomial of degree n with positive leading coefficient together with $(*)$ determines (P_n) uniquely. It can be calculated from the moments s_n via the formula

$$P_n(x) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ s_1 & s_2 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad D_n = \det \mathcal{H}_n.$$

Characterization of indeterminacy (Hamburger)

The following conditions are equivalent:

(i) Indeterminacy

(ii) $\sum_0^\infty |P_n(i)|^2 < \infty$

(iii) $P^2(z) := \sum_0^\infty |P_n(z)|^2 < \infty$ for all $z \in \mathbb{C}$.

In (iii) the series converges uniformly on compact subsets of $\mathbb{C}$.
The moment problems corresponding to the classical orthogonal polynomial systems: Hermite, Laguerre, Jacobi, Legendre, Chebyshev are determinate.

If μ has compact support it is determinate.

Stieltjes (1894) gave the first examples of indeterminate measures, e.g. the lognormal distribution in statistics. The orthogonal polynomials are called Stieltjes-Wigert polynomials.

The classical treatises of the moment problem

Hamburger's theorem and the spectral theorem for self-adjoint operators in Hilbert space are closely related. One can deduce each of the theorems from the other.

Marshall Stone: *Linear transformations in Hilbert space*, 1932 treats the moment problem as an application of the theory of self-adjoint extensions of symmetric operators.

Monographs:

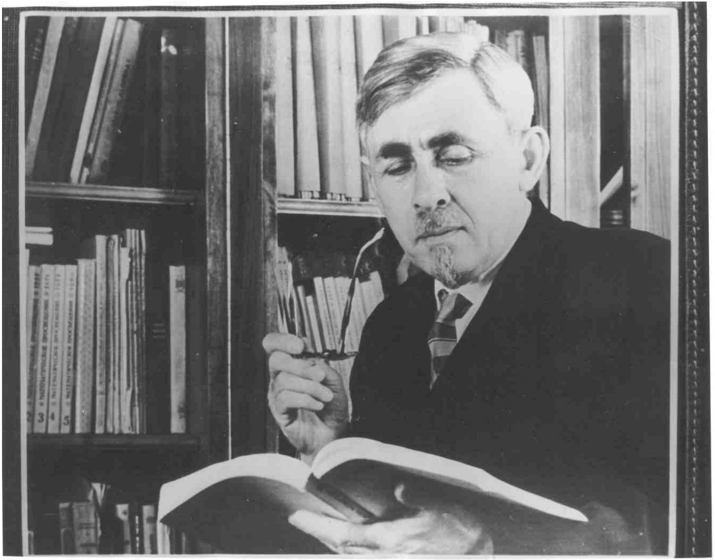
J. Shohat, J.D. Tamarkin, *The Problem of Moments*, 1943

N.I. Akhiezer: *The classical moment problem*, 1965 (Russian edition 1961).

New treatment:

B. Simon, *The classical moment problem as a self-adjoint finite difference operator*. Adv. Math. **137** (1998), 82–203.

My mathematical hero: N.I. Akhiezer (1901-1980)



A lower bound for λ_∞ in the indeterminate case

In the indeterminate case

$$\lambda_\infty \geq \left(\frac{1}{2\pi} \int_0^{2\pi} P^2(e^{i\theta})\, d\theta \right)^{-1} > 0,$$

where

$$P^2(z) = \sum_{n=0}^{\infty} |P_n(z)|^2, \quad z \in \mathbb{C}.$$

So far there are no formulas expressing λ_∞ by other known quantities.

Indeterminate case: The log-normal moments

$0 < q < 1$: log-normal moments are $s_n = q^{-n(n+2)/2}$ given by

$$\frac{\sqrt{q}}{\sqrt{2\pi \log(1/q)}} \int_0^\infty x^n \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right)\, dx.$$

Defining

$$h(x) = \sin(\frac{2\pi}{\log(1/q)} \log x)$$

then the non-negative densities $(-1 \leq r \leq 1)$

$$\frac{\sqrt{q}}{\sqrt{2\pi \log(1/q)}} \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) [1 + rh(x)]$$

and the discrete measures $(a > 0)$

$$\frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{k(k+2)/2} \delta_{aq^k}$$

all have the log-normal moments.

Correspondence between Stieltjes and Hermite

Stieltjes to Hermite: January 30, 1892

"L'existence de ces fonctions $\varphi(x)$ qui, sans être nulles, sont telles que

$$\int_0^\infty x^n \varphi(x)\, dx = 0, \quad n = 0, 1, \ldots,$$

me paraît très remarquable"

$$\varphi(x) = \sin\left(\frac{2\pi}{\log(1/q)} \log x\right) \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right)$$

is one of these functions.

The behaviour of λ_N in some classical cases

In case of Hermite and Laguerre polynomials Szegő (1936) found the asymptotic behaviour

$$\lambda_N \sim AN^{1/4}B^{\sqrt{N}}$$

for certain explicit values $A > 0, 0 < B < 1$. Szegő also found the asymptotics for Legendre polynomials and this was generalized by Widom and Wilf (1966) for measures μ in the so-called Szegő class meaning that μ has a density $w(x)$ on a compact interval $[a, b]$ satisfiying

$$\int_a^b \frac{\log w(x)}{\sqrt{(x-a)(b-x)}}\, dx > -\infty.$$

The behaviour is

$$\lambda_N \sim AN^{1/2}B^N,$$

where A, B are certain constants as above.

Orthogonal polynomials and Favard's theorem

The orthonormal polynomials (P_n) satisfy a three-term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x), n \geq 0, \quad (3\text{trl})$$

where $a_n \in \mathbb{R}$ and $b_n > 0$.
Conversely, given two sequences $(a_n), (b_n)$ of respectively real and positive numbers, then (3trl) together with the initial conditions $P_{-1}(x) = 0, P_0(x) = 1$ determine polynomials P_n of degree n and they are orthonormal with respect to one or several measures μ with infinite support.

This is Favard's Theorem (1935)

The spectral theory of Jacobi matrices

The study of orthonormal polynomials is equivalent to study symmetric Jacobi matrices defined in terms of $a_n \in \mathbb{R}$ and $b_n > 0$

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \cdots \\ b_0 & a_1 & b_1 & 0 & \cdots \\ 0 & b_1 & a_2 & b_2 & \cdots \\ 0 & 0 & b_2 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

acting as a densely defined symmetric operator in ℓ^2 with the standard orthonormal basis (e_n). The moments can be calculated from J via and $s_n = (J^n e_0, e_0)$.

J has either deficiency indices $(0,0)$ (the determinate case) or $(1,1)$ (the indeterminate case).

Work by B and Szwarc (2011) about λ_N

From this work I shall mention 3 results.
The first one is that one has exponential decay to 0 in case of compact support.

Theorem (Exponential decay)

Suppose (b_n) *bounded and* $b := \limsup b_n$. *Then*

$$\limsup \lambda_N^{1/N} \le \frac{2b^2}{1+2b^2}$$

and hence

$$\lambda_N \le AB^N, \quad \textit{for suitable } A > 0, 0 < B < 1.$$

In particular this holds for any measure μ *with compact support.*

Work by B and Szwarc (2011) — slow and fast decay

The eigenvalues λ_N for a determinate moment problem can decay to zero arbitrarily slow or arbitrarily fast:

Theorem (slow decay)

Let (τ_n) *be a decreasing sequence of positive numbers satisfying* $\tau_n \to 0$ *and* $\tau_0 < 1/2$*. Then there exists a determinate symmetric probability measure* μ *on* $\mathbb{R}$ *for which* $\lambda_N \geq \tau_N$ *for all* N*.*

Theorem (fast decay)

Let (τ_n) *be a decreasing sequence of positive numbers satisfying* $\tau_n \to 0$ *and* $\tau_0 = 1$*. Then there exists a determinate symmetric probability measures* μ *for which* $\lambda_N \leq \tau_N$ *for all* N*.*

The inverses of the Hankel matrices $\mathcal{H}_N$ (Aitken(1939))

Define the coefficient matrix

$$\mathcal{A}_N = \left(a_{j,l}^{(N)}\right)_{j,l=0}^{N}, \quad K_N(x,y) = \sum_{n=0}^{N} P_n(x)P_n(y) = \sum_{j,l=0}^{N} a_{j,l}^{(N)} x^j y^l.$$

Then $\mathcal{A}_N = \mathcal{H}_N^{-1}$. The proof is very simple: For $0 \leq k \leq N$ we have by the reproducing property

$$\int x^k K_N(x,y)\, d\mu(x) = y^k.$$

On the other hand we have

$$\int x^k K_N(x,y)\, d\mu(x) = \sum_{l=0}^{N} \left(\sum_{j=0}^{N} s_{k+j} a_{j,l}^{(N)} \right) y^l,$$

hence

$$\sum_{j=0}^{N} s_{k+j} a_{j,l}^{(N)} = \delta_{k,l}.$$

A digression

For Lebesgue measure on $[0,1]$ we have

$$s_n = \int_0^1 x^n \, dx = \frac{1}{n+1}$$

so $\mathcal{H}_N$ is the Hilbert matrix $\{1/(j+k+1)\}$. The orthogonal polynomials are Legendre polynomials

$$p_n(x) = \frac{1}{n!} D^n [x(1-x)]^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n} x^k$$

The orthonormal Legendre polynomials are $P_n(x) = (-1)^n \sqrt{2n+1} p_n(x)$ so

$$K_N(x,y) = \sum_{n=0}^{N} (2n+1) p_n(x) p_n(y)$$

has integer coefficients, i.e., $\mathcal{H}_N^{-1}$ has integer entries.

Formal inverse of the infinite Hankel matrix $\mathcal{H}$

In the indeterminate case we know that

$$\sum_{n=0}^{\infty} |P_n(x)|^2 < \infty, \quad x \in \mathbb{C}$$

so we can consider the entire function on $\mathbb{C}^2$

$$K(x,y) = \sum_{n=0}^{\infty} P_n(x) P_n(y) = \sum_{j,l=0}^{\infty} a_{j,l} x^j y^l.$$

It is easy to see that $a_{j,l}^{(N)} \to a_{j,l}$ for $N \to \infty$.
Define the infinite symmetric matrix $\mathcal{A} = \{a_{j,l}\}$.
Natural question:

Is $\mathcal{HA} = \mathcal{AH} = \mathcal{I}$?
Since $\mathcal{H}$ and $\mathcal{A}$ are symmetric it is enough to verify $\mathcal{AH} = \mathcal{I}$.

Preliminary comments to the question

By a famous result of Carleman:

$$\text{Indeterminacy} \quad \Longrightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}}} < \infty,$$

so $\sqrt[2n]{s_{2n}} \to \infty$, i.e., $s_{2n} \to \infty$ quite fast and $s_{2n} \geq 1$ for $n \geq n_0$.
Therefore

$$\sum_{j=0}^{\infty} s_{j+k}^2 = \infty, \quad k = 0, 1, \ldots.$$

so $\mathcal{H}$ does not define an operator on ℓ^2
Question: Is it true that

$$\sum_{k=0}^{\infty} |s_{j+k} a_{k,l}| < \infty, \text{ for all } j, l \geq 0?$$

If this is the case we say that $\mathcal{AH}$ is absolutely convergent ?

Some answers

1: The answer is yes for Stieltjes-Wigert polynomials, i.e. $s_n = q^{-n(n+2)/2}, 0 < q < 1$ (B-Szwarc 2011).
2: For any indeterminate moment problem the matrix $\mathcal{A}$ is of trace class and

$$\sum_{j,l=0}^{\infty} |a_{j,l}|^{\varepsilon} < \infty \quad \text{for any} \quad \varepsilon > 0.$$

3: $\mathcal{A}$ is positive definite. Let $c_n = \sqrt{a_{n,n}}$. Then $\lim n \sqrt[n]{c_n} = 0$ so

$$\Phi(z) = \sum_{n=0}^{\infty} c_n z^n$$

is an entire function of minimal exponential type. Its order and type are equal to the order and type of the indeterminate moment problem, which is by definition the order and type of the entire function $zK(z,0)$ (and of several other functions associated with the indeterminate moment problem).

Some auxiliary matrices $\mathcal{B}, \mathcal{C}$

We use the following notation for the orthonormal polynomials

$$\begin{aligned} P_n(x) &= b_{n,n}x^n + b_{n-1,n}x^{n-1} + \ldots + b_{1,n}x + b_{0,n}, \\ x^n &= c_{n,n}P_n(x) + c_{n-1,n}P_{n-1}(x) + \ldots + c_{1,n}P_1(x) + c_{0,n}P_0(x). \end{aligned}$$

By the three-term recurrence relation we get

$$b_{n,n} = \frac{1}{b_0 b_1 \ldots b_{n-1}}, \qquad c_{n,n} = b_0 b_1 \ldots b_{n-1}. \tag{2}$$

The matrices $\mathcal{B} = \{b_{i,j}\}$ and $\mathcal{C} = \{c_{i,j}\}$ with the assumption

$$b_{i,j} = c_{i,j} = 0 \qquad \text{for } i > j$$

are upper-triangular. Since $\mathcal{B}$ and $\mathcal{C}$ are transition matrices between two sequences of linearly independent systems of functions, we have

$$\mathcal{BC} = \mathcal{CB} = \mathcal{I}.$$

The relation to $\mathcal{H}$ and $\mathcal{A}$: $\mathcal{H} = \mathcal{C}^t\mathcal{C} \quad \mathcal{A} = \mathcal{B}\mathcal{B}^t$

$s_{m+n} = \langle x^m, x^n \rangle_{L^2(\mu)} = c_{0,m}c_{0,n} + c_{1,m}c_{1,n} + \ldots + c_{m,m}c_{m,n},\ m \leq n$
so $\mathcal{H} = \mathcal{C}^t\mathcal{C}.$
Inserting $P_n(x) = b_{n,n}x^n + b_{n-1,n}x^{n-1} + \ldots + b_{1,n}x + b_{0,n}$ in

$$K(x,y) = \sum_{n=0}^{\infty} P_n(x)P_n(y) = \sum_{j,l=0}^{\infty} a_{j,l}x^j y^l$$

$$K(x,y) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} b_{j,n}x^j \right) \left(\sum_{l=0}^{n} b_{l,n}y^l \right)$$

hence (formally)

$$a_{j,l} = \sum_{n=\max(j,l)}^{\infty} b_{j,n}b_{l,n}, \quad \mathcal{A} = \mathcal{B}\mathcal{B}^t.$$

Rigorously: $\mathcal{B}$ is Hilbert-Schmidt, hence $\mathcal{A}$ is trace class.

The order of certain indeterminate moment problems

Consider a symmetric indeterminate moment problem given by $b_n > 0$ and $a_n = 0$ for all n.

Definition (b_n) is called regular if it is either eventually log-convex, i.e., there exists $n_0 \in \mathbb{N}$ such that

$$b_n^2 \le b_{n-1} b_{n+1}, \quad n \ge n_0, \tag{3}$$

or eventually log-concave, i.e., the reverse inequality holds.

Theorem (B and Szwarc, Adv. Math. 2014)

Let (b_n) be a regular sequence of positive numbers. Then the corresponding symmetric moment problem is indeterminate if and only if $\sum 1/b_n < \infty$. In the indeterminate case the order ρ of the moment problem is equal to the exponent of convergence of (b_n)

$$\mathcal{E}(b_n) = \inf \left\{ \alpha > 0 \mid \sum_{n=0}^{\infty} \frac{1}{b_n^\alpha} < \infty \right\}.$$

Work in progress with Szwarc, Wrocław

Theorem

Assume that (b_n) is eventually log-convex and strictly increasing. The corresponding moment problem is indeterminate of order 0. The matrix product $\mathcal{A}\mathcal{H}$ is absolutely convergent and $\mathcal{A}\mathcal{H} = \mathcal{I}$.

Examples: $b_n = exp(an^b)$, with $a > 0, b \geq 1$.

Theorem

Assume that $b_n = (n+1)^c, c > 0$. Then b_n is log-concave and the symmetric moment problem is indeterminate if and only if $c > 1$.
The order is $1/c$.
The product $\mathcal{A}\mathcal{H}$ is absolutely convergent for $c > 3/2$ with $\mathcal{A}\mathcal{H} = \mathcal{I}$.
There exists a symmetric moment problem of order $2/3$ such that $\mathcal{A}\mathcal{H}$ is not absolutely convergent.

Ideas of proof I

We study two quantities:

$$U_n := \frac{s_{2n}}{b_0^2 b_1^2 \ldots b_{n-1}^2}, \quad V_n = b_0 b_1 \ldots b_{n-1} c_n,$$

where we recall that

$$c_n = \sqrt{a_{n,n}} = \left(\sum_{j=n}^{\infty} b_{n,j}^2 \right)^{1/2}.$$

Both quantities are scale invariant: If (b_n) is replaced by (λb_n) for some $\lambda > 0$, then U_n, V_n are unchanged.

Ideas of proof II

In the symmetric case the odd moments vanish and
$s_{2n} = (J^{2n}e_0, e_0)$, say

$$s_0 = 1, s_2 = b_0^2, s_4 = b_0^2(b_0^2 + b_1^2), s_6 = (b_0 b_1 b_2)^2 + b_0^2 b_1^4 + 2b_0^4 b_1^2 + b_0^6$$

but these polynomials in $b_0, b_1, \ldots$ are difficult to control.
We write

$$\frac{x^n}{b_0 b_1 \ldots b_{n-1}} = \sum_{k=0}^{[n/2]} u_{n,k} P_{n-2k}(x)$$

hence by Parseval

$$U_n = \sum_{k=0}^{[n/2]} u_{n,k}^2.$$

Note that $u_{n,0} = 1$ and

$$u_{n+1,k} = \frac{b_{n-2k}}{b_n} u_{n,k} + \frac{b_{n+1-2k}}{b_n} u_{n,k-1}, \quad 1 \le k \le [n/2].$$

Ideas of proof III

One can get estimates of $u_{n,k}$ and then U_n via this recurrence. Note that in the log-convex case b_{n-1}/b_n is decreasing, while in the log-concave case it is increasing, and this permits estimates.
Example:

Theorem

Assume (b_n) is log-convex and strictly increasing. Define $q = b_0/b_1 < 1$. Then

$$U_n \le (q^2;q^2)_\infty^{-2} \sum_{k=0}^{\infty} q^{2k^2}.$$

Here we have used the notation from the theory of basic hypergeometric functions: For $z \in \mathbb{C}$, $0 < q < 1$ we write

$$(z;q)_n = \prod_{j=0}^{n-1} \left(1 - zq^j\right), \; n = 1, 2, \ldots, \infty, \quad (z;q)_0 = 1.$$

Some references

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C. Berg and R. Szwarc, *Inverse of infinite Hankel moment matrices*. In preparation.

Thank you for your attention

An illustration

We shall prove that for all $n = 0, 1, 2, \dots$

$$I_n := \int_0^\infty x^n \exp\left(-\frac{(\log x)^2}{2\log(1/q)}\right) \sin(\frac{2\pi}{\log(1/q)} \log x)\, dx = 0.$$

Substitution $u = \log x$ and writing $c := \log(1/q) > 0$

$$I_n := \int_{-\infty}^{\infty} \exp(u(n+1)) \exp\left(-\frac{u^2}{2c}\right) \sin(\frac{2\pi}{c} u)\, du.$$

Using

$$\frac{(u-(n+1)c)^2}{2c} = \frac{u^2}{2c} - (n+1)u + \frac{c}{2}(n+1)^2$$

and translation invariance of the integral we get

$$I_n = \exp((c/2)(n+1)^2) \int_{-\infty}^{\infty} \exp(-\frac{u^2}{2c}) \sin\left(\frac{2\pi}{c}(u+(n+1)c)\right)\, du = 0.$$

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Moments of the discrete distribution

We shall prove that

$$s_n = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} (aq^k)^n a^k q^{k(k+2)/2} = q^{-n(n+2)/2}$$

where

$$L(a) = s_0 = \sum_{k=-\infty}^{\infty} a^k q^{k(k+2)/2}.$$

Write

$$q^{k^2/2+kn} = q^{(k+n)^2/2-n^2/2}$$

to get

$$s_n = \frac{q^{-n^2/2}}{L(a)} \sum_{k=-\infty}^{\infty} a^{k+n} q^k q^{(k+n)^2/2} = q^{-n-n^2/2}$$

using translation invariance of the sum.

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