Hankel Moore-Penrose Condition Numbers

Nikolai Nikolski

Institute of Mathematics, Bordeaux

and

Chebyshev Laboratory, St.Petersburg

I. Hankel Operators on the Hardy space H^2

Hankel: a bounded linear operator $\Gamma: H^2 \longrightarrow H^2$,

$$H^2 = \{f: f = \sum_{k\geq 0} \hat{f}(k) z^k, \ \sum_{k\geq 0} |\hat{f}(k)|^2 = \|f\|^2 < \infty\}$$

having a matrix

$$\Gamma = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \dots & \dots \\ c_1 & c_2 & c_3 & c_4 & \dots & \dots \\ c_2 & c_3 & c_4 & c_5 & \dots & \dots \\ c_3 & c_4 & c_5 & c_6 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

with respect to the standard basis $(z^k)_0^\infty$, or equivalently

$$\Gamma S = S^* \Gamma,$$

where $Sf = zf(z)$ stands for the shift operator on H^2.

• **Notation** $\Gamma = \Gamma_f$ **if** $c_k = \hat{f}(k)$, $k \geq 0$ **(Fourier coefficients of** $f \in L^2(\mathbb{T})$**)**
• **Nehari's theorem:** Γ **is bounded on** $H^2 \Leftrightarrow \Gamma = \Gamma_f$ **with** $f \in L^\infty(\mathbb{T})$ **and**

$$\|\Gamma\| = min\{\|f\|_\infty : \Gamma = \Gamma_f, f \in L^\infty(\mathbb{T})\} = \|f\|_{L^\infty/H^\infty_-}\}$$

where $H^\infty_- = \{f : f \in L^\infty(\mathbb{T}), \hat{f}(k) = 0 \ for \ k \geq 0\}$**.**

• **Yet another useful model for Hankel operators:** $H = \mathcal{J}\Gamma : H^2 \longrightarrow H^2_-$, **where** $H^2_- = L^2 \ominus H^2 = \{f : f \in L^2(\mathbb{T}), \hat{f}(k) = 0 \ for \ k \geq 0\}$ **and** $\mathcal{J}f = \overline{z}f(\overline{z})$ $(z \in \mathbb{T})$, $f \in L^2(\mathbb{T})$; $\mathcal{J}$ **is a unitary symmetry on** L^2, $\mathcal{J}z^k = z^{-k-1}$ $(k \in \mathbb{Z})$, $\mathcal{J}^2 = I$**.**
• **Nehari's theorem: a Hankel** $H : H^2 \longrightarrow H^2_-$ **is bounded** $\Leftrightarrow \exists g \in L^\infty(\mathbb{T})$ **s.t.**

$$Hx = H_g x =: P_-(gx) \ (x \in H^2),$$ P_- **is the orthoprojection on** H^2_-**.**

• **In fact,** $\|H_g\| = \|g\|_{L^\infty/H^\infty}$ **and** $\mathcal{J}\Gamma_f = H_{\mathcal{J}f}$**.**

Condition Numbers

• Condition number of a linear operator $CN(A) = \|A\| \cdot \|A^{-1}\|$

• Important everywhere where the size of inverses $\|A^{-1}\|$ or the resolvents $\|(\lambda I - A)^{-1}\|$ matters:

- for (effective) similarity $V^{-1}AV$, $CN(V) \leq$ ...;

- for functional calculi $f(A) = \frac{1}{2\pi i} \int_\gamma (\lambda I - A)^{-1} f(\lambda) d\lambda$;

- for computational linear algebra and $n \times n$ matrices A:

- as the maximal relative errors for perturbed equations: $A(x + \Delta x) = y + \Delta y$,

$$CN(A) = max_{x,\Delta y} \frac{\|\Delta x\| / \|x\|}{\|\Delta y\| / \|y\|},$$

- as a measure for the linear independence of columns $(Ae_k)_{k=0}^{n-1}$:

$$\frac{1}{CN(A)} = min\{\frac{\|A - B\|}{\|A\|} : rank(B) < n\}.$$

• Condition number $CN(\Gamma)$ has no sense for a Hankel Γ (always $0 \in \sigma(\Gamma)$).

But Γ can be invertible if we disregard the kernel $Ker\Gamma$.

• Inverses disregarding the kernels are called Moore-Penrose inverses: A and B are (mutually) Penrose inverse to each other if $BAB = B$, $ABA = A$ (and AB, BA selfadjoint). It is to say, $AB = I$ on the $Range(A)$, $BA = I$ on the range B, complemented by 0 on complements.

• Fact: The restriction $\Gamma|(Ker\Gamma)^{\perp}$ can be invertible, and moreover $|\Gamma|\ |(Ker\Gamma)^{\perp}$ is an arbitrary positive operator, up to unitary equivalence (S.Treil, 1990/1991).

• Moreover, $KerH_f = Ker\Gamma_{\mathcal{J}f} \neq \{0\} \Leftrightarrow$ there exist $\varphi \in H^\infty$ and a Beurling inner function Θ ($\Theta \in H^\infty$ and $|\Theta| = 1$ $a.e.$ on $\mathbb{T}$) such that $KerH_f = \Theta H^2$, and $H_f = H_{\overline{\Theta}\varphi}$.

• **Let** $H : \ H^2 \longrightarrow \ H^2_-$ **be a Hankel having a kernel,**

$$KerH = \ \Theta H^2, \Theta \textbf{ inner, } H = \ H_{\overline{\Theta}\varphi}, \varphi \in H^\infty,$$

and let $K_\Theta = \ (KerH)^\perp = \ H^2 \ominus \Theta H^2$ **the so-called "model space" and** $M_\Theta : K_\theta \longrightarrow \ K_\Theta$ **the model operator,**

$$M_\Theta x = \ P_\Theta(zx), \ x \in K_\Theta,$$

P_Θ **stands for the orthogonal projection on** K_Θ**.**

• **D.Clark, 1972:** $\Theta H_{\overline{\Theta}\varphi} = \ \varphi(M_\Theta)P_\Theta$**, and hence (**$H_{\overline{\Theta}\varphi}$ **is Penrose invertible)** $\Leftrightarrow$ **(**$Range(H_{\overline{\Theta}\varphi})$ **closed)** $\Leftrightarrow \varphi(M_\Theta)$ **invertible** $\Leftrightarrow$ **A Bezout equation** $\varphi h + \ \Theta g = \ 1$ **is solvable in** $h, g \in H^\infty$**, and moreover**

$$\|H^{(-1)}_{\overline{\Theta}\varphi}\| = \ \|\varphi(M_\Theta)^{-1}\| = \ min\|h\|_\infty,$$

the inf **(**min**) is taken over all these solutions.**

• **Comments: The quantity** $inf\|h\|_\infty$, $\varphi h + \Theta g = 1$, **could be estimated in terms of** $inf_{z\in\mathbb{D}}(|\varphi(z)|^2 + |\Theta(z)|^2)$, **but usually the latter it is not available...**

The available quantity is

$$\delta_\varphi =: \ inf\{|\varphi(z)| : \ z \in \sigma(M_\Theta)\} \ \textbf{(if } \varphi \in H^\infty \cap C(\overline{\mathbb{D}})\textbf{)},$$

or even only $\delta_\varphi = \ inf\{|\varphi(\lambda)| : |\lambda| < \ 1, \ \Theta(\lambda) = \ 0\}$.

• **The set** $\{\lambda : |\lambda| < 1, \ \Theta(\lambda) = 0\} = \ \sigma_p(M_\Theta)$ **is the point spectrum of** M_Θ**, the reproducing kernels** $k_\lambda(z) = \ \frac{1}{1-\overline{\lambda}z}$, $\Theta(\lambda) = \ 0$, **are still eigenvectors of** $\varphi(M_\Theta)^*$:

$$\varphi(M_\Theta)^* k_\lambda = \ \overline{\varphi(\lambda)} k_\lambda, \ \Theta(\lambda) = \ 0.$$

• **The problem is whether there exists a function** $t \longmapsto \ c(t)$, $t > 0$ **such that**

$$\|H^{(-1)}_{\overline{\Theta}\varphi}\| = \ \|\varphi(M_\Theta)^{-1}\| \leq \ c(\delta_\varphi), \ \forall \varphi \in H^\infty \ ?$$

• **Given an inner function** Θ **we define** $c(\delta) =$

$= sup\{\|H^{(-1)}_{\overline{\Theta}\varphi}\| = \|\varphi(M_\Theta)^{-1}\| : \ \delta \leq \ \delta_\varphi = inf_{\sigma(\Theta)}|\varphi| \leq \ \|\varphi\|_\infty \leq \ 1\}$,

where $0 < \delta < 1$, $\sigma(\Theta) = \ \sigma(M_\Theta) = \ \{z : \ |z| \leq \ 1, \ \underline{lim}_{\zeta \longrightarrow z}|\Theta(\zeta)| = 0\}$ **and** $\varphi \in H^\infty \cap C(\overline{\mathbb{D}})$ **(the disc algebra).**

• **Comments: (1) Normalization** $\|\varphi\|_\infty \leq 1$ **is necessary to have an estimate for condition numbers** $CN(H_{\overline{\Theta}\varphi}) = \ CN(\varphi(M_\Theta))$.

(2) In fact, the estimate given by $c(\delta)$ **can be written directly in** CN**-terms, as follows**

$$\frac{1}{\Delta(\varphi)} \leq \ CN(H_{\overline{\Theta}\varphi}) = \ CN(\varphi(M_\Theta)) \leq \ c(\frac{1}{\Delta(\varphi)}),$$

(sharp estimates) where $\Delta(\varphi) = \ r(\varphi(M_\Theta)^{-1})\|\varphi(M_\Theta)\|$ **is a** "SPECTRAL CONDITION NUMBER" **(the norm** $\|\varphi(M_\Theta)^{-1}\|$ **is replaced by the spectral radius** $r(\varphi(M_\Theta)^{-1})$**).**

(3) The problem is to decide whether $c(\delta) < \ \infty$ **for all (certain)** $0 < \delta < 1$.

• **More notation:**

• **The pseudohyperbolic distance between** $z, w \in \mathbb{D}$ **is** $\rho(z, w) = |b_z(w)|$, **where** $b_z(w) = \frac{z-w}{1-\bar{z}w}$ **stands for a Blaschke factor.**
• **An inner function** Θ **can be factored into** $\Theta = BS$ **where**

$$B = \Pi_{k\geq 1} b_{\lambda_k} \text{ and } S = exp(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\nu(\zeta))$$

are, respectively, a Blaschke product over the zeroes $Z(\Theta) = (\lambda_k)_{k\geq 1}$ **of** Θ **in the disc** $\mathbb{D}$ **and a singular inner function,** $\nu \geq 0$ **being a singular Borel measure on** $\mathbb{T}$.
• **A Borel measure** $\mu \geq 0$ **on the disc** $\mathbb{D}$ **is said to be a "Carleson measure" if** $H^2 \subset L^2(\mu)$; μ **is Carleson if and only if**

$$sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} d\mu(\zeta) < \infty$$

(the Reproducing Kernel Test).
• **A measure associated with a Blaschke product** B **is defined as** $\mu_B = \Sigma_{k\geq 1}(1-|\lambda_k|^2)\delta_{\lambda_k}$.

• THEOREM 1 (P.Gorkin, R.Mortini, N.N. -2008). **Given an inner function Θ, with the above notation, the following properties are equivalent.**

(1) $\forall\delta,\ 0 < \delta < 1 \ \Rightarrow\ c(\delta) < \ \infty$.

(2) If $\varphi \in H^\infty$ **and** $\delta_\varphi = \ inf_{\Theta(\lambda)=0}|\varphi(\lambda)| > \ 0$ **then** $\varphi(M_\Theta)$ **is invertible (**$H_{\overline{\Theta}_\varphi}$ **Penrose invertible).**

(3) $\Theta = \ BS$, **and** $\forall\epsilon > 0\ \exists\eta > 0$ **s.t.** $\{|S| \le \eta\} \subset \ \{|B| \le \epsilon\}$ **and** μ_B **is a "Weak Carleson measure":** $\forall\epsilon > 0$

$$sup_{\rho(w,Z(\Theta))\ge\epsilon} \int_{\mathbb{D}} \frac{1-|w|^2}{|1-\overline{w}\zeta|^2} d\mu_B(\zeta) < \infty.$$

(4) $\forall\epsilon > 0\ \eta(\epsilon) =: \ inf\{|\Theta(w)| : \ \rho(w, Z(\Theta)) \ge \ \epsilon\} \ > \ 0$.

Moreover, $c(\delta) \le \ \frac{a}{\eta(\delta/3)^2} log \frac{1}{\eta(\delta/3)}$ **for every** δ**,** $0 < \delta < 1$ $(a > 0$ **is a numerical constant), and so**

$$\|H^{(-1)}_{\overline{\Theta}_\varphi}\| = \|\varphi(M_\Theta)^{-1}\| \le c(\delta_\varphi)\ (\forall\varphi \in H^\infty,\ \|\varphi\|_\infty \le \ 1).$$

• **Comments: (a) If $\sigma(M_\Theta)$ is in a Stolz angle then (3) $\Leftrightarrow$ (3')** $\Theta = BS$, $S = 1$ **and** μ_B **is a Carleson measure (not only "weak Carleson").**
(b) In the latter case (3'), $Z(B) = (\lambda_k)_{k\geq 1}$ **is a finite union of interpolating sequences (say,** N**) and**

$$c(\delta) \leq \frac{a}{\delta^{2N}} log\frac{e}{\delta}, \ 0 < \delta < 1.$$

(c) In general, $\delta \longmapsto c(\delta)$ **is a non-increasing function on** $(0, 1)$ **which can be infinite for some** δ**,** $\delta \in (0, \delta(\Theta))$**, and finite for** $\delta \in (\delta(\Theta), 1)$**. For every** $\delta_1 \in [0, 1]$ **there exists** $\Theta = B$ **such that** $\delta(B) = \delta_1$ **(Vasyunin + N., 2011).**
(d) Even if $\delta(\Theta) = 0$**,** $c(\delta)$ **can grow arbitrarily fast as** $\delta \downarrow 0$ **(Borichev, 2013).**
(e) In fact, $\delta(\Theta) = inf\{\epsilon > 0 : \ \eta(\epsilon) > 0\}$ **(Borichev-Nicolau-Thomas, 2017).**

II. Cripto-Hankel Integral Operators

- "(Almost) every operator is Hankel"

Below $A : H \longrightarrow H$ is a bounded Hilbert space operator.

- Every non-negative operator $A \geq 0$ having $0 \in \sigma_{ess}(A)$ and $dimKerA \in \{0, \infty\}$ is the modulus $|\Gamma|$ of a Hankel Γ with respect to an orthonormal basis (S.Treil, 1990).

- Every A with $0 \in \sigma_{ess}(A)$ and $dimKerA \in \{0, \infty\}$, being multiplied by a unitary operator, has a Hankel matrix Γ with respect to an orthonormal basis (is a "cripto-hankel operator").

  In particular, $CN(A) = CN(\Gamma)$.

- Every selfadjoint operator with simple spectrum has a Hankel matrix with respect to an orthonormal basis (A.Megretsky- V.Peller- S.Treil, 1995).

An example: lower triangular integral operators

• Let μ be Borel probability measure on $[0,1]$ and J_μ *an integration operator*

$$J_\mu f(x) = \int_{[0,x>} f d\mu,\ 0 \leq x \leq 1,$$

on the spaces $L^p([0,1],\mu)$.

• $[0,x>$ can be $[0,x)$ or $[0,x]$, or - which is better for a symmetry reason between J_μ and J_μ^* - an arithmetic mean of these two:

$$J_\mu f(x) = \int_{[0,x>} f d\mu = \int_{[0,x)} f d\mu + \tfrac{1}{2}\mu(\{x\})f(x),\ x \in [0,1].$$

• We use the standard decomposition of μ, $\mu = \mu_c + \mu_d$, in the discrete $\mu_d = \Sigma_{y\in[0,1]}\, \mu(\{y\})\delta_y$ and the continuous components. If $\mu_d = 0$, then

$$J_\mu f(x) = \int_0^x f d\mu.$$

• $J_\mu : L^p(\mu) \longrightarrow L^p(\mu)$, $1 \leq p \leq \infty$, is *a compact operator* whose spectrum

$$\sigma(J_\mu : L^p(\mu) \longrightarrow L^p(\mu))$$

does not depend on p and consists of $\{0\}$ and the *eigenvalues* $\frac{1}{2}\mu(\{y\})$, $y \in [0,1]$.

• **Consider the algebra of lower triangular integral operators generated by J_μ,**

$$A_{\mu,p} = \ alg_{L^p(\mu)}(J_\mu),$$

the norm closure of polynomials in $J_\mu : L^p(\mu) \longrightarrow L^p(\mu)$**,** $1 \leq p \leq \infty$, $J^0_\mu =: \ id.$

• **We will bounding condition numbers of operators in $A_{\mu,p}$ in terms of the spectral condition numbers.**

The question is treated as the well/ill-posedness of the inversion problem in $A_{\mu,p}$, in the following sense.

• **The problem (as before) is to find a bound** $CN(S) \leq c(1/\Delta(S))$ **in terms of the spectral condition number** $\Delta(S) = r(S^{-1})\|S\|$**,** $S \in A_{\mu,p}$**.**

• **Define**

$$\delta_S = min(|\lambda| : \lambda \in \sigma(S)), \text{ where } S \in A_{\mu,p},$$

$$c(\delta) = sup\{\|S^{-1}\| : \delta \leq \delta_S \leq \|S\| \leq 1, \ S \in A_{\mu,p}\}, 0 < \delta \leq 1,$$

$$\delta(A_{\mu,p}) = inf\{\delta \in (0,1] : c(\delta) < \infty\},$$

(a "*critical constant*": $c(\delta) = \infty$ for $0 < \delta < \delta(A_{\mu,p})$, and $c(\delta) < \infty$ for $\delta(A_{\mu,p}) < \delta \leq 1$).

• **Comment: this is a kind of the well/ill-posedness of the inversion problem for polynomials in** $J_{\mu,p}$**:**

- **well-posed if** $\delta(A_{\mu,p}) = 0$**, and**
- **ill-posed if** $\delta(A_{\mu,p}) > 0$ (... there exists an "invisible" but numerically detectable spectrum).

• *Today, I can manage the problem for two following cases only:*
- $p = 1$ or ∞ AND $\mu = \mu_c$ (continuous measure),
- $p = 2$, μ arbitrary.

• We say that *a sequence of positive numbers* $(a_n)_{n\geq 1}$ *geometrically decrease* if $sup_{n\geq 1}\frac{a_{n+1}}{a_n} < 1$.

• **Theorem 1.** *For the case* $p = 1$, $\mu = \mu_c$, *we have*

$$\delta(A) = 1/2, \textbf{ and } c(\delta) = \frac{1}{2\delta - 1} \textbf{ for } 1/2 < \delta \leq 1.$$

• **Theorem 2.** *For the case* $p = 2$, *the following alternative holds.*

(1) Either, $\mu_c = 0$ and $\sigma(J_\mu)$ *is a (finite) union of* N *geometrically decreasing sequences, and then*

$$\delta(A_{\mu,2}) = 0 \text{ and } c(\delta) \leq a\frac{log\frac{1}{\delta}}{\delta^{2N}},\ 0 < \delta < 1,$$

where $a > 0$ *depends on* N *and ratios of geometric sequences in* $\sigma(J_\mu)$.

(2) Or, this is not the case, and then $\delta(A_{\mu,2}) = 1$ *(so that,* $c(\delta) = \infty$ *for every* $0 < \delta < 1$*).*

- **Hints to the proof of Theorem 2:**

1) The operator $J_\mu : L^2(\mu) \longrightarrow L^2(\mu)$ has a nonnegative real part:

$$J_\mu^* = \int_{<x,1]} f d\mu, 2Re(J_\mu)f = \int_{[0,1]} f d\mu = (f,1)_{L^2(\mu)}1,\ f \in L^2(\mu),$$

and $rank(Re(J_\mu)) = 1$.

Consequently, its *Cayley transform* C_μ is a contraction,

$$C_\mu =: (I - J_\mu)(I + J_\mu)^{-1},\ \|C_\mu\| \leq 1,$$

having *rank 1 defects*, $rank(I - C_\mu^* C_\mu) = rank(I - C_\mu C_\mu^*) = 1$.

2) $alg(J_\mu) = alg(C_\mu)$, $\sigma(C_\mu) = \omega(\sigma(J_\mu)) \subset [0,1]$ where $\omega(z) = (1-z)(1+z)^{-1}$.

3) C_μ **is unitarily equivalent** to its *functional model* $M_\Theta : K_\Theta \longrightarrow K_\Theta$ where $\Theta = \theta_\mu$ stands for the *characteristic function* of C_μ.

• **Hints to the proof of Theorem 2 (cont'd/end):**

4) Computing the characteristic function,

$$\theta_\mu(z) = ((I + i\sqrt{2Re(J_\mu)}(J_\mu^* - zI)^{-1}\sqrt{2Re(J_\mu)})1, 1)_{L^2(\mu)}\|1\|^{-2},$$

$$\theta_\mu(z) = \prod_{k\geq 1} b_{\lambda_k}(z) \cdot exp(-\mu_c([0,1])\frac{1+z}{1-z}),$$

where $\lambda_k = \dfrac{1 - \dfrac{\mu(\{x_k\})}{2}}{1 + \dfrac{\mu(\{x_k\})}{2}}$ are eigenvalues of C_μ, $b_{\lambda_k}(z) = \frac{\lambda_k - z}{1-\bar{\lambda}_k z}$ an elementary Blaschke factor.

5) Applying the above GMN theorem we get the result. ∎

• References

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The End

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Thank you!