Numerical ranges of restricted shifts, and norms of truncated Toeplitz operators

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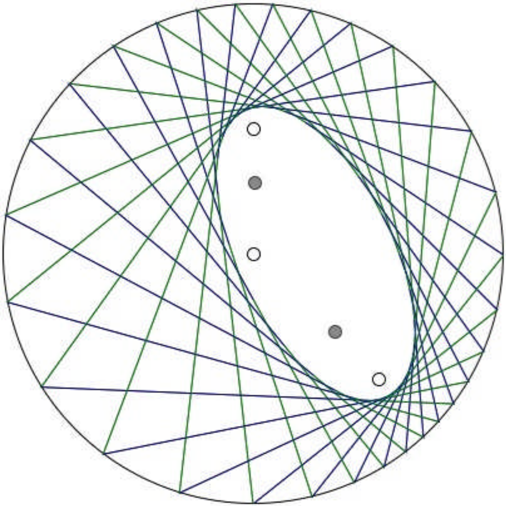
Joint work with Pamela Gorkin (Bucknell) et al

A Poncelet ellipse



What can this possibly have to do with Hankel operators?

Hardy spaces

As usual $H^2(\mathbb{D})$ denotes the Hardy space of the unit disc $\mathbb{D}$, the functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

It embeds isometrically as a subspace of $L^2(\mathbb{T})$, with $\mathbb{T}$ the unit circle,

$$f(e^{it}) \sim \sum_{n=0}^{\infty} a_n e^{int}.$$

Orthogonal decomposition

Indeed we may write

$$L^2(\mathbb{T}) = H^2 \oplus (H^2)^\perp,$$

so that

$$\sum_{n=-\infty}^{\infty} a_n e^{int} = \sum_{n=0}^{\infty} a_n e^{int} + \sum_{n=-\infty}^{-1} a_n e^{int},$$

and

$$f \in H^2 \iff \overline{z}\overline{f} \in (H^2)^\perp = \overline{H_0^2}.$$

Here, and often from now on, we write $z = e^{it}$.

Toeplitz operators in brief

For $g \in L^\infty(\mathbb{T})$ we define the Toeplitz operator T_g on H^2 by

$$T_g f = P_{H^2}(gf) \qquad (f \in H^2),$$

or multiplication followed by orthogonal projection. It is well known that $\|T_g\| = \|g\|_\infty$, and if g has Fourier coefficients (c_n), then T_g has the matrix

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \ddots \\ c_2 & c_1 & c_0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Inner–outer factorizations

Recall that if $f \in H^2$, not the 0 function, then it has an inner–outer factorization (unique up to unimodular constants)

$$f = \theta u$$

with θ inner, i.e., $|\theta(e^{it})| = 1$ a.e., and with u outer (no nontrivial inner divisors). Equivalently, u is outer when

$$\overline{\mathrm{span}}(u, zu, z^2u, \ldots) = H^2.$$

Model spaces

The factorization follows from Beurling's theorem, which says that the non-trivial closed invariant subspaces for the shift $S = T_z$ are the subspaces θH^2, with θ inner.

Now it follows that the invariant subspaces for the backwards shift $S^* = T_{\overline{z}}$ are the **model spaces**

$$K_\theta = H^2 \ominus \theta H^2 = H^2 \cap \theta \overline{H_0^2}$$

with θ inner.

It is easy to check that $K_\theta = \ker T_{\overline{\theta}}$.

Examples

(i) Take $\theta(z) = z^n$, and then

$$K_\theta = \mathrm{span}(1, z, z^2, \dots, z^{n-1}).$$

(ii) Take

$$\theta(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z},$$

a finite Blaschke product with distinct zeroes $a_1, a_2, \dots, a_n$ in $\mathbb{D}$. Then

$$K_\theta = \mathrm{span}\left(\frac{1}{1 - \overline{a_1} z}, \dots, \frac{1}{1 - \overline{a_n} z} \right).$$

The restricted shift

We write

$$S_\theta = P_{K_\theta} S_{|K_\theta},$$

for the adjoint of the restriction of S^* to its invariant subspace K_θ.

More generally for suitable $g \in L^\infty(\mathbb{T})$ the *truncated Toeplitz operator* with symbol g is

$$A_g^\theta = P_{K_\theta} M_g,$$

where M_g is multiplication by g. So $S_\theta = A_z^\theta$.

Unitary perturbations

Suppose $\theta(0) = 0$. Then D.N. Clark (1972) parametrised the unitary rank-1 perturbations of S_θ as $\{U_\alpha : \alpha \in \mathbb{T}\}$, where

$$U_\alpha f = S_\theta f + \alpha \langle f, S^* \theta \rangle 1 \qquad (f \in K_\theta).$$

Alternatively for arbitrary θ we may look at unitary Halmos 1-dilations on $K_\theta \oplus \mathbb{C}$, which are

$$U = \begin{pmatrix} S_\theta & * \\ * & * \end{pmatrix}$$

and in 1-1 correspondence with perturbations of $S_{z\theta}$.

Example

For $\theta(z) = z^n$, the operator S_θ has matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

using the basis $\{1, z, z^2, \dots, z^{n-1}\}$, and the unitary perturbations are

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \alpha \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $|\alpha| = 1$.

Numerical ranges

Recall that for a Hilbert space operator we have

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\},$$

a convex set (and closed if $\dim H < \infty$) with

$$\sigma(T) \subseteq \overline{W(T)}.$$

Part of this talk is devoted to understanding the numerical range of S_θ.
Note that $\sigma(S_\theta)$ contains the zeroes of θ.

Other properties of the numerical range

Functional calculus property:

$$\operatorname{Re} W(T) \leq c \iff \|\exp(tT)\| \leq \exp(ct) \quad (t \geq 0).$$

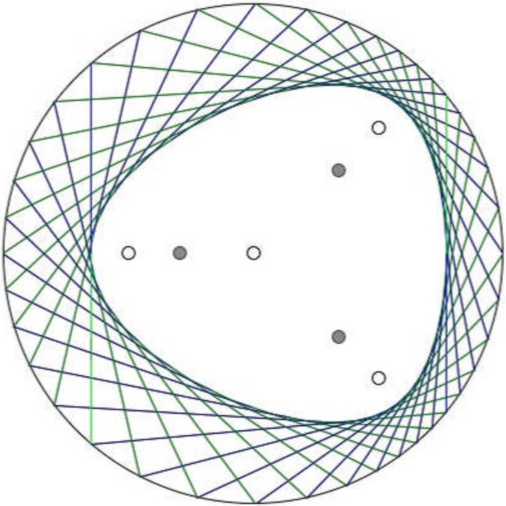
Crouzeix conjecture:

$$\|f(T)\| \leq 2 \sup\{|f(z)| : z \in W(T)\},$$

known to be true with a worse constant.

For $f(z) = z$ the best constant is certainly 2.

Poncelet generalized



Numerical ranges of S_θ and U_α (quadrilaterals) for θ a Blaschke of degree 3.

Theorems about numerical ranges

Gau and Wu (1998) for finite Blaschke products θ.

$$W(S_\theta) = \bigcap_{\alpha \in \mathbb{T}} W(U_\alpha),$$

an intersection of closed polygons.

Chalendar, Gorkin, JRP (2009). For all inner functions θ

$$\overline{W(S_\theta)} = \bigcap_{\alpha \in \mathbb{T}} \overline{W(U_\alpha)}.$$

Earlier and later work

Choi and Li (2001), the Halmos conjecture,

$$\overline{W(T)} = \bigcap_U \overline{W(U)},$$

where U is a unitary dilation on $H \oplus H$.

Benhida, Gorkin, Timotin (2011), and then Bercovici and Timotin (2014). For completely non-unitary contractions with defect indices n,

$$\overline{W(T)} = \bigcap_U \overline{W(U)},$$

where U is a unitary dilation of T on $H \oplus \mathbb{C}^n$.

Those polygons

The vertices of the polygons are solutions to $zB(z) = \lambda$ for $\lambda \in \mathbb{T}$. Also if

$$\frac{B(z)}{zB(z) - \lambda} = \sum_{j=1}^{n+1} \frac{m_j}{z - z_j},$$

then each $m_j > 0$ and the points of tangency are

$$\frac{m_{j+1} z_j + m_j z_{j+1}}{m_j + m_{j+1}} \qquad \text{(Gau and Wu (2004)).}$$

Used by Chalendar–Gorkin–JRP–Ross (2016) to decide when inner functions can be factorized under composition.

Interpolation

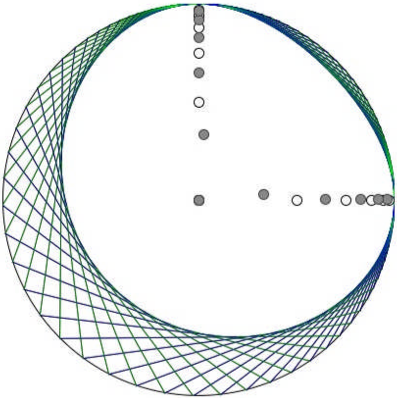
The function $zB(z)$ maps $\mathbb{T}$ to itself with an $(n+1)$ to 1 cover.
How many polygons do we need to determine the numerical range? Answer: 2.

If two Blaschkes θ and φ of degree $n+1$ identify two sets $\{z_1, \ldots, z_{n+1}\}$ and $\{w_1, \ldots, w_{n+1}\}$ of the circle (so that θ and φ are constant on both sets), then they are Frostman shifts of each other,

$$\varphi = \lambda \frac{\theta - a}{1 - \overline{a}\theta},$$

and if both vanish at 0 then they have the same zeroes (Chalendar-Gorkin-JRP, 2011).

Infinitely-many zeroes



Numerical ranges of S_θ and U_α for θ an infinite Blaschke product, zeroes accumulating at 1 and i.

From numerical ranges to norms

Lumer's 1961 theorem asserts that

$$\max\{\mathrm{Re}\,\lambda : \lambda \in W(T)\} = \lim_{a \to 0+} \frac{1}{a}\{\|I + aT\| - 1\}.$$

Clearly, by replacing T by $e^{it}T$, we may find the numerical radius in different directions.

Thus we want to study $\|I + aS_B\|$ for $a \in \mathbb{C}$, small.

There is another version, using $\exp(aS_B)$, but this is less useful here.

A link with Hankel operators

For an analytic truncated Toeplitz operator

$$A_g^\theta = P_{K_\theta} M_g,$$

with $g \in H^\infty$, we have

$$\|A_g^\theta\| = \mathrm{dist}(\overline{\theta}g, H^\infty) = \|\Gamma_{\overline{\theta}g}\|$$

with $\Gamma_f : H^2 \to \overline{H_0^2}$ the Hankel operator

$$\Gamma_f u = P_{\overline{H_0^2}}(fu).$$

Calculating the norm by interpolation

Best illustrated with θ a finite Blaschke product B. A problem much studied by analysts and engineers.

The Nevanlinna-Pick approach goes by interpolation. For $\|1 + aS_B\| \leq \gamma$ precisely when we can solve

$$1 + az = B(z)g(z) + \gamma h(z)$$

with $g, h \in H^\infty$ and $\|h\|_\infty \leq \gamma$.

Equivalently, $h(z_k) = 1 + az_k$, where $z_1, \ldots, z_n$ are the zeroes of B.

The Pick matrix

Solution of the interpolation problem is possible if and only if the matrix with (j, k) entry

$$\frac{1 - (1 + az_j)(1 + \overline{az_k})/\gamma^2}{1 - z_j \overline{z}_k}$$

is positive semi-definite.

This can be used to show that for real zeroes the numerical radius of S_B is attained on the real axis.

The Foias–Tannenbaum approach (1987)

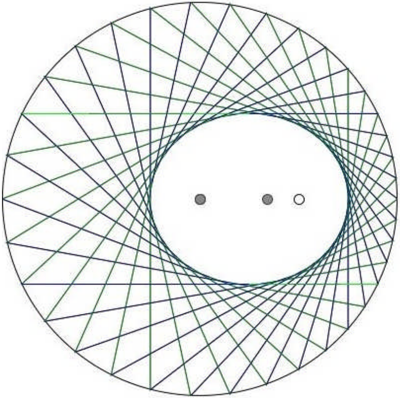
Take $|a| < 1$ and for $\rho > 0$ let

$$P_\rho = I - \frac{1}{4\rho^2}(1 + aS_B)(1 + \bar{a}S_B^*)$$

and the largest ρ for which P_ρ is singular is the norm of $(I + aS_B)/2$.

Some ingenious calculations (theirs!) make this a practical way of obtaining information on the numerical radius.

A very easy example



$$B(z) = z\left(\frac{z-1/2}{1-z/2}\right).$$

$W(S_B)$ an ellipse, foci 0 and $1/2$, major axis $[-\frac{1}{4}, \frac{3}{4}]$.

Vertical tangents at $zB(z) = \pm 1$.

$\|I + aS_B\| = 1 + \frac{3}{4}a + o(a)$ for $a > 0$.

Norms of Truncated Toeplitz operators

More generally, we now look at the norm of the TTO A_g^θ, where θ is inner and $g \in L^\infty$.

Theorem (Garcia and Ross) Suppose that θ is not a finite Blaschke product, and ξ is a limit point of its zeroes. If g is continuous on an open arc containing ξ with $|g(\xi)| = \|g\|_\infty$, then $\|A_g^\theta\| = \|g\|_\infty$.

Note that for $g \in H^\infty$, this is also giving us the norm of the Hankel operator $\Gamma_{\overline{\theta}g}$.

A more general result

The previous result becomes much simpler if one uses Banach algebra ideas.

Let $M(H^\infty)$ be the maximal ideal space of a Banach algebra, and $Z(\theta)$ the zeroes of an inner function θ in $M(H^\infty)$.

Proposition (Gorkin–JRP, 2017). Suppose θ is inner and not invertible in $H^\infty + C(\mathbb{T})$. For $f \in L^\infty$, if $\hat{f}(x) = \|f\|_\infty$ for some $x \in Z(u)$, then

$$\mathrm{dist}(f, \theta H^\infty) = \|f\|_\infty.$$

Compact operators without continuous symbols I

Theorem (Bessonov). Let θ be inner and $g \in H^\infty + C(\mathbb{T})$.Then A_g^θ is compact if and only if $g \in \theta(H^\infty + C(\mathbb{T}))$.

Chalendar, Fricain and Timotin (in a survey article) ask for an example of a compact TTO with symbol in $\theta(H^\infty + C(\mathbb{T}))$ that possesses no continuous symbol.

We do this next.

Compact operators without continuous symbols II

Example (Gorkin–JRP). Let B be an interpolating Blaschke product with zero sequence (z_n) clustering at every point of $\mathbb{T}$.

Let $f \in H^\infty + C(\mathbb{T})$ with $f(z_n) \to 0$ but $f(z_n) \neq 0$ for all n (for example, f could be another Blaschke product with nearby zeroes).

Then A_f^B is compact, but has no continuous symbol.

Truncated Hankel operators (THO)

For θ inner and a symbol g we may define the truncated Hankel operator

$$B_g^\theta : K_\theta \to \overline{zK_\theta}, \qquad B_g^\theta(f) = P_{\overline{zK_\theta}}(gf).$$

Since $\overline{zK_\theta} = \overline{\theta} K_\theta$, we have in fact

$$B_g^\theta(f) = \overline{\theta} A_{\theta g}^\theta(f),$$

as observed by Bessonov.

Multiplication by $\overline{\theta}$ is a unitary map from K_θ onto $\overline{zK_\theta}$, thus the results on norms and compactness of TTO have natural analogues for THO.

The end

That's all. Thank you.