On the essential norms of Toeplitz operators

Eugene Shargorodsky

Department of Mathematics
King's College London

Fredholm operators and the essential spectrum

For Banach spaces X and Y, let $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ denote the sets of bounded linear and compact linear operators from X to Y, respectively.

For $A \in \mathcal{B}(X, Y)$, let

$$\operatorname{Ker} A := \{x \in X |\ Ax = 0\}, \quad \operatorname{Ran} A := \{Ax |\ x \in X\}.$$

The operator A is called Fredholm if

$$\dim \operatorname{Ker} A < +\infty, \ \ \dim (X / \operatorname{Ran} A) < +\infty.$$

Fredholm operators and the essential spectrum

For Banach spaces X and Y, let $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ denote the sets of bounded linear and compact linear operators from X to Y, respectively.

For $A \in \mathcal{B}(X, Y)$, let

$$\operatorname{Ker} A := \{x \in X |\ Ax = 0\}, \quad \operatorname{Ran} A := \{Ax |\ x \in X\}.$$

The operator A is called Fredholm if

$$\dim \operatorname{Ker} A < +\infty, \ \ \dim (X/\operatorname{Ran} A) < +\infty.$$

The essential spectrum of $A \in \mathcal{B}(X) := \mathcal{B}(X, X)$ is the set

$$\operatorname{Spec}_{\mathrm{e}}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not Fredholm}\}.$$

Essential norm

The essential norm of $A \in \mathcal{B}(X, Y)$ is defined as follows

$$\|A\|_{\mathrm{e}} := \inf\{\|A - K\| : \ K \in \mathcal{K}(X, Y)\}.$$

Essential norm

The essential norm of $A \in \mathcal{B}(X, Y)$ is defined as follows

$$\|A\|_{\mathrm{e}} := \inf\{\|A - K\| : K \in \mathcal{K}(X, Y)\}.$$

For any $A \in \mathcal{B}(X)$, $\mathrm{Spec}_{\mathrm{e}}(A)$ and $\|A\|_{\mathrm{e}}$ are equal to the spectrum and the norm of the corresponding element $[A]$ of the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$.

Essential norm

The essential norm of $A \in \mathcal{B}(X, Y)$ is defined as follows

$$\|A\|_{\mathrm{e}} := \inf\{\|A - K\| : \ K \in \mathcal{K}(X, Y)\}.$$

For any $A \in \mathcal{B}(X)$, $\mathrm{Spec}_{\mathrm{e}}(A)$ and $\|A\|_{\mathrm{e}}$ are equal to the spectrum and the norm of the corresponding element $[A]$ of the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$.

$$r_{\mathrm{e}}(A) := \sup\{|\lambda| : \ \lambda \in \mathrm{Spec}_{\mathrm{e}}(A)\} \leq \|A\|_{\mathrm{e}}.$$

Toeplitz operators

Hardy space:

$$H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) \big| \ f_n = 0 \ \text{ for } n < 0\}, \quad 1 \leq p \leq \infty,$$

where f_n is the n-th Fourier coefficient of f.

Toeplitz operators

Hardy space:

$$H^p(\mathbb{T}) := \{f \in L^p(\mathbb{T}) \big| \ f_n = 0 \ \text{ for } n < 0\}, \quad 1 \le p \le \infty,$$

where f_n is the n-th Fourier coefficient of f.

Toeplitz operator generated by a function $a \in L^\infty(\mathbb{T})$:

$$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}), \ \ 1 < p < \infty,$$
$$T(a)f = P(af), \quad f \in H^p(\mathbb{T}),$$

where P is the Riesz projection:

$$P\left(\sum_{n=-\infty}^{+\infty} g_n \zeta^n\right) = \sum_{n=0}^{+\infty} g_n \zeta^n, \quad \zeta \in \mathbb{T}.$$

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

P. Hartman and A. Wintner (1954, $p = 2$),
I.B. Simonenko (1968)

$$a(\mathbb{T}) \subseteq \mathrm{Spec}_e(T(a))$$

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

P. Hartman and A. Wintner (1954, $p = 2$),
I.B. Simonenko (1968)

$$a(\mathbb{T}) \subseteq \mathrm{Spec_e}(T(a))$$

Hence,

$$\|a\|_{L^\infty} \le r_\mathrm{e}(T(a)) \le \|T(a)\|_\mathrm{e}$$

P. Hartman and A. Wintner (1954, $p = 2$),
I.B. Simonenko (1968)

$$a(\mathbb{T}) \subseteq \mathrm{Spec}_{\mathrm{e}}(T(a))$$

Hence,

$$\|a\|_{L^\infty} \leq r_{\mathrm{e}}(T(a)) \leq \|T(a)\|_{\mathrm{e}}$$

On the other hand,

$$\|T(a)\|_{\mathrm{e}} \leq \|T(a)\| = \|Pa\mathrm{I}\| \leq \|P\|\|a\|_{L^\infty}$$

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

B. Hollenbeck and I.E. Verbitsky (2000):

$$\|P\|_{L^p \to L^p} = \frac{1}{\sin \frac{\pi}{p}}$$

B. Hollenbeck and I.E. Verbitsky (2000):

$$\|P\|_{L^p \to L^p} = \frac{1}{\sin \frac{\pi}{p}}$$

This is considerably more difficult to prove than the result (essentially) due to S.K. Pichorides (1972)

$$\|S\|_{L^p \to L^p} = \max \left\{ \tan \frac{\pi}{2p}, \ \cot \frac{\pi}{2p} \right\},$$

where $S = 2P - \mathrm{I}$ is the Cauchy singular integral operator.

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

B. Hollenbeck and I.E. Verbitsky (2000):

$$\|P\|_{L^p \to L^p} = \frac{1}{\sin \frac{\pi}{p}}$$

This is considerably more difficult to prove than the result (essentially) due to S.K. Pichorides (1972)

$$\|S\|_{L^p \to L^p} = \max \left\{ \tan \frac{\pi}{2p},\ \cot \frac{\pi}{2p} \right\},$$

where $S = 2P - \mathrm{I}$ is the Cauchy singular integral operator.

Putting together:

$$\|a\|_{L^\infty} \le \|T(a)\|_{\mathrm{e}} \le \frac{1}{\sin \frac{\pi}{p}} \|a\|_{L^\infty}$$

Example. Let

$$a_0\left(e^{i\vartheta}\right) := \frac{1}{\sin\frac{\pi}{p}} - \cot\left(\frac{\pi}{p}\right) e^{\pm i\pi/p}, \quad \pm\vartheta \in (0,\pi).$$

Then $\|a_0\|_{L^\infty} = 1$.

Example. Let

$$a_0\left(e^{i\vartheta}\right) := \frac{1}{\sin\frac{\pi}{p}} - \cot\left(\frac{\pi}{p}\right)e^{\pm i\pi/p}, \quad \pm\vartheta \in (0,\pi).$$

Then $\|a_0\|_{L^\infty} = 1$.

The Fredholm theory of Toeplitz operators with piecewise continuous symbols (Gohberg-Krupnik) $\Longrightarrow$

$$\frac{1}{\sin\frac{\pi}{p}} \in \mathrm{Spec}_\mathrm{e}(T(a_0))$$

Example. Let

$$a_0\left(e^{i\vartheta}\right) := \frac{1}{\sin\frac{\pi}{p}} - \cot\left(\frac{\pi}{p}\right) e^{\pm i\pi/p}, \quad \pm\vartheta \in (0,\pi).$$

Then $\|a_0\|_{L^\infty} = 1$.

The Fredholm theory of Toeplitz operators with piecewise continuous symbols (Gohberg-Krupnik) $\Longrightarrow$

$$\frac{1}{\sin\frac{\pi}{p}} \in \mathrm{Spec}_{\mathrm{e}}(T(a_0)) \implies \|T(a_0)\|_{\mathrm{e}} \geq \frac{1}{\sin\frac{\pi}{p}}$$

Example. Let

$$a_0\left(e^{i\vartheta}\right) := \frac{1}{\sin\frac{\pi}{p}} - \cot\left(\frac{\pi}{p}\right) e^{\pm i\pi/p}, \quad \pm\vartheta \in (0, \pi).$$

Then $\|a_0\|_{L^\infty} = 1$.

The Fredholm theory of Toeplitz operators with piecewise continuous symbols (Gohberg-Krupnik) $\Longrightarrow$

$$\frac{1}{\sin\frac{\pi}{p}} \in \mathrm{Spec}_{\mathrm{e}}(T(a_0)) \implies \|T(a_0)\|_{\mathrm{e}} \geq \frac{1}{\sin\frac{\pi}{p}}$$

Hence the constant $\frac{1}{\sin\frac{\pi}{p}}$ is optimal in

$$\|a\|_{L^\infty} \leq \|T(a)\|_{\mathrm{e}} \leq \frac{1}{\sin\frac{\pi}{p}}\|a\|_{L^\infty}, \quad \forall a \in L^\infty(\mathbb{T}).$$

Toeplitz operators with continuous symbols

Consider $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ with $a \in C(\mathbb{T})$.

I. Gohberg (1952), ...

$$\mathrm{Spec}_{\mathrm{e}}(T(a)) = a(\mathbb{T}).$$

In particular, $\mathrm{Spec}_{\mathrm{e}}(T(a))$ **does not** depend on p.

Toeplitz operators with continuous symbols

Consider $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$, $1 < p < \infty$ with $a \in C(\mathbb{T})$.

I. Gohberg (1952), ...

$$\mathrm{Spec}_e(T(a)) = a(\mathbb{T}).$$

In particular, $\mathrm{Spec}_e(T(a))$ **does not** depend on p.

A. Böttcher, N. Krupnik, and B. Silbermann (1988): Does $\|T(a)\|_e$ depend on p if $a \in C(\mathbb{T})$? Is it true that

$$\|T(a)\|_e = \|a\|_{L^\infty}, \quad \forall a \in C(\mathbb{T})?$$

Notation

$$\mathbf{e}_m(z) := z^m\,, \quad z \in \mathbb{C}, \;\; m \in \mathbb{Z}.$$

Notation

$$\mathbf{e}_m(z) := z^m\,, \quad z \in \mathbb{C}, \;\; m \in \mathbb{Z}.$$

A. Böttcher, N. Krupnik, and B. Silbermann (1988):

$$\|T(a)\|_{\mathrm{e}} = \|a\|_{L^\infty}, \quad \forall a \in (C + H^\infty)(\mathbb{T})$$
$$\iff \|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = 1.$$

Notation

$$\mathbf{e}_m(z) := z^m\,, \quad z \in \mathbb{C}, \ \ m \in \mathbb{Z}.$$

A. Böttcher, N. Krupnik, and B. Silbermann (1988):

$$\|T(a)\|_{\mathrm{e}} = \|a\|_{L^\infty}, \quad \forall a \in (C + H^\infty)(\mathbb{T})$$
$$\iff \|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = 1.$$

So, the question is whether or not the last equality holds.

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

A. Böttcher, N. Krupnik, and B. Silbermann (1988):

$$\|T(\mathbf{e}_{-1})\| > 1, \quad \forall p \in (1, \infty) \setminus \{2\}$$

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

A. Böttcher, N. Krupnik, and B. Silbermann (1988):

$$\|T(\mathbf{e}_{-1})\| > 1, \quad \forall p \in (1, \infty) \setminus \{2\}$$

Theorem 2

$$\|T(a)\|_{\mathrm{e}} \le 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty} \le 2\|a\|_{L^\infty}, \qquad \forall a \in (C + H^\infty)(\mathbb{T}), \quad \forall p \in (1, \infty).$$

Measures of noncompactness of a linear operator

Let Y be a Banach space. For a bounded subset Ω of Y, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r.

Measures of noncompactness of a linear operator

Let Y be a Banach space. For a bounded subset Ω of Y, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r.
For $A \in \mathcal{B}(X, Y)$, set

$$\|A\|_\chi := \chi\left(A(B_X)\right),$$

where B_X denotes the unit ball in X.

Measures of noncompactness of a linear operator

Let Y be a Banach space. For a bounded subset Ω of Y, we denote by $\chi(\Omega)$ the greatest lower bound of the set of numbers r such that Ω can be covered by a finite family of open balls of radius r.
For $A \in \mathcal{B}(X, Y)$, set

$$\|A\|_\chi := \chi\left(A(B_X)\right),$$

where B_X denotes the unit ball in X.

Let $\|A\|_m$ denote the greatest lower bound of all numbers η having the property that there exists a subspace M of X having finite codimension and such that

$$\|Ax\| \leq \eta\|x\|, \quad \forall x \in M.$$

A. Lebow and M. Schechter (1971):

$$\|A\|_\chi/2 \leq \|A\|_m \leq 2\|A\|_\chi$$

and

$$\|A\|_\chi \leq \|A\|_{\mathrm{e}}\,, \quad \|A\|_m \leq \|A\|_{\mathrm{e}}$$

A Banach space Y is said to have the bounded compact approximation property (BCAP) if there exists a constant $M \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $F \subset Y$, there exists an operator $T \in \mathcal{K}(Y)$ such that $\|\mathrm{I} - T\| \leq M$ and

$$\|y - Ty\| < \varepsilon, \ \ \forall y \in F.$$

A Banach space Y is said to have the bounded compact approximation property (BCAP) if there exists a constant $M \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $F \subset Y$, there exists an operator $T \in \mathcal{K}(Y)$ such that $\|\mathrm{I} - T\| \leq M$ and

$$\|y - Ty\| < \varepsilon, \quad \forall y \in F.$$

We say that Y has the dual compact approximation property (DCAP) if there exists a constant $M^* \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $G \subset Y^*$, there exists an operator $T \in \mathcal{K}(Y)$ such that $\|\mathrm{I} - T\| \leq M^*$ and

$$\|z - T^* z\| < \varepsilon, \quad \forall z \in G.$$

Approximation properties of Banach spaces

A Banach space Y is said to have the bounded compact approximation property (BCAP) if there exists a constant $M \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $F \subset Y$, there exists an operator $T \in \mathcal{K}(Y)$ such that $\|\mathrm{I} - T\| \leq M$ and

$$\|y - Ty\| < \varepsilon, \quad \forall y \in F.$$

We say that Y has the dual compact approximation property (DCAP) if there exists a constant $M^* \in (0, +\infty)$ such that given any $\varepsilon > 0$ and any finite set $G \subset Y^*$, there exists an operator $T \in \mathcal{K}(Y)$ such that $\|\mathrm{I} - T\| \leq M^*$ and

$$\|z - T^* z\| < \varepsilon, \quad \forall z \in G.$$

We denote by $M(Y)$ and $M^*(Y)$ the infima of the constants M and M^* for which the above conditions are satisfied.

A. Lebow and M. Schechter (1971): If Y has the BCAP, then

$$\|A\|_{\mathrm{e}} \leq M(Y)\|A\|_{\chi}\,, \quad \forall A \in \mathcal{B}(X, Y)$$

and hence

$$\|A\|_{\mathrm{e}} \leq 2M(Y)\|A\|_{m}\,, \quad \forall A \in \mathcal{B}(X, Y).$$

A. Lebow and M. Schechter (1971): If Y has the BCAP, then

$$\|A\|_{\mathrm{e}} \leq M(Y)\|A\|_{\chi}, \quad \forall A \in \mathcal{B}(X, Y)$$

and hence

$$\|A\|_{\mathrm{e}} \leq 2M(Y)\|A\|_{m}, \quad \forall A \in \mathcal{B}(X, Y).$$

(K. Astala and H.-O. Tylli (1987): If $\|A\|_{\mathrm{e}} < M\|A\|_{\chi}$ for every $A \in \mathcal{B}(X, Y) \setminus \mathcal{K}(X, Y)$ and every Banach space X, then Y has the BCAP and $M(Y) \leq M$.)

A. Lebow and M. Schechter (1971): If Y has the BCAP, then

$$\|A\|_{\mathrm{e}} \leq M(Y)\|A\|_{\chi}\,, \quad \forall A \in \mathcal{B}(X,Y)$$

and hence

$$\|A\|_{\mathrm{e}} \leq 2M(Y)\|A\|_{m}\,, \quad \forall A \in \mathcal{B}(X,Y).$$

(K. Astala and H.-O. Tylli (1987): If $\|A\|_{\mathrm{e}} < M\|A\|_{\chi}$ for every $A \in \mathcal{B}(X,Y) \setminus \mathcal{K}(X,Y)$ and every Banach space X, then Y has the BCAP and $M(Y) \leq M$.)

Theorem

If X has the DCAP, then

$$\|A\|_{\mathrm{e}} \leq M^{*}(X)\|A\|_{m}\,, \quad \forall A \in \mathcal{B}(X,Y).$$

Approximation properties of Hardy spaces

Theorem

The Hardy space $H^p = H^p(\mathbb{T})$, $1 < p < \infty$ has the bounded compact approximation and the dual compact approximation properties with

$$M(H^p), M^*(H^p) \leq 2^{\left|1-\frac{2}{p}\right|}.$$

The proof of the above theorem uses the Fejér means

$$
\begin{aligned}
(\mathbf{K}_n f)\left(e^{i\vartheta}\right) &:= (K_n * f)\left(e^{i\vartheta}\right) = \int_{-\pi}^{\pi} K_n\left(e^{i\vartheta - i\theta}\right) f\left(e^{i\theta}\right)\, d\theta, \\
K_n\left(e^{i\theta}\right) &:= \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} \\
&= \frac{1}{2\pi(n+1)} \left(\frac{\sin \frac{(n+1)\theta}{2}}{\sin \frac{\theta}{2}}\right)^2, \\
& \qquad \vartheta, \theta \in [-\pi, \pi], \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

The proof of the above theorem uses the Fejér means

$$\begin{aligned}
(\mathbf{K}_n f)\left(e^{i\vartheta}\right) &:= (K_n * f)\left(e^{i\vartheta}\right) = \int_{-\pi}^{\pi} K_n\left(e^{i\vartheta - i\theta}\right) f\left(e^{i\theta}\right)\, d\theta, \\
K_n\left(e^{i\theta}\right) &:= \frac{1}{2\pi}\sum_{k=-n}^{n}\left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} \\
&= \frac{1}{2\pi(n+1)}\left(\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\right)^2, \\
&\quad \vartheta, \theta \in [-\pi, \pi], \quad n = 0, 1, 2, \ldots
\end{aligned}$$

$$\|\mathbf{K}_n\|_{L^p \to L^p} = 1 \implies \|\mathrm{I} - \mathbf{K}_n\|_{L^p \to L^p} \le 2 \quad \text{for } 1 \le p \le \infty.$$

The proof of the above theorem uses the Fejér means

$$\begin{aligned}
(\mathbf{K}_n f)\left(e^{i\vartheta}\right) &:= (K_n * f)\left(e^{i\vartheta}\right) = \int_{-\pi}^{\pi} K_n\left(e^{i\vartheta - i\theta}\right) f\left(e^{i\theta}\right)\, d\theta, \\
K_n\left(e^{i\theta}\right) &:= \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} \\
&= \frac{1}{2\pi(n+1)} \left(\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\right)^2, \\
& \qquad \vartheta, \theta \in [-\pi, \pi], \quad n = 0, 1, 2, \ldots
\end{aligned}$$

$\|\mathbf{K}_n\|_{L^p \to L^p} = 1 \implies \|\mathrm{I} - \mathbf{K}_n\|_{L^p \to L^p} \le 2$ for $1 \le p \le \infty$.
Parseval's theorem $\implies \|\mathrm{I} - \mathbf{K}_n\|_{L^2 \to L^2} = 1$.

The proof of the above theorem uses the Fejér means

$$
\begin{aligned}
(\mathbf{K}_n f)\left(e^{i\vartheta}\right) &:= (K_n * f)\left(e^{i\vartheta}\right) = \int_{-\pi}^{\pi} K_n\left(e^{i\vartheta - i\theta}\right) f\left(e^{i\theta}\right)\, d\theta, \\
K_n\left(e^{i\theta}\right) &:= \frac{1}{2\pi} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta} \\
&= \frac{1}{2\pi(n+1)} \left(\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\right)^2, \\
& \vartheta, \theta \in [-\pi, \pi], \quad n = 0, 1, 2, \ldots
\end{aligned}
$$

$\|\mathbf{K}_n\|_{L^p \to L^p} = 1 \implies \|\mathrm{I} - \mathbf{K}_n\|_{L^p \to L^p} \leq 2$ for $1 \leq p \leq \infty$.
Parseval's theorem $\implies \|\mathrm{I} - \mathbf{K}_n\|_{L^2 \to L^2} = 1$.
Interpolation $\implies \|\mathrm{I} - \mathbf{K}_n\|_{L^p \to L^p} \leq 2^{\left|1-\frac{2}{p}\right|}$.

Reminder:

Theorem 2

$$\|T(a)\|_{\mathrm{e}} \leq 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty} \leq 2\|a\|_{L^\infty}, \qquad \forall a \in (C+H^\infty)(\mathbb{T}), \quad \forall p \in (1,\infty).$$

Reminder:

Theorem 2

$$\|T(a)\|_{\mathrm{e}} \le 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty} \le 2\|a\|_{L^\infty}, \qquad \forall a \in (C+H^\infty)(\mathbb{T}), \ \forall p \in (1,\infty).$$

According to the above results, it is sufficient to show that

$$\|T(a)\|_m = \|a\|_{L^\infty}.$$

Reminder:

Theorem 2

$$\|T(a)\|_{\mathrm{e}} \le 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty} \le 2\|a\|_{L^\infty}, \qquad \forall a \in (C+H^\infty)(\mathbb{T}),\ \forall p \in (1,\infty).$$

According to the above results, it is sufficient to show that

$$\|T(a)\|_m = \|a\|_{L^\infty}.$$

The latter is easy to prove if $a = \mathbf{e}_{-n}h$, $h \in H^\infty(\mathbb{T})$.

Reminder:

Theorem 2

$$\|T(a)\|_{\mathrm{e}} \le 2^{\left|1-\frac{2}{p}\right|}\|a\|_{L^\infty} \le 2\|a\|_{L^\infty}, \qquad \forall a \in (C+H^\infty)(\mathbb{T}), \ \forall p \in (1,\infty).$$

According to the above results, it is sufficient to show that

$$\|T(a)\|_m = \|a\|_{L^\infty}.$$

The latter is easy to prove if $a = \mathbf{e}_{-n}h$, $h \in H^\infty(\mathbb{T})$.
Such functions are dense in $(C+H^\infty)(\mathbb{T})$.

Reminder:

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

Reminder:

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

It is sufficient to show that

$$\|T(\mathbf{e}_{-1})\|_{\chi} \geq \|T(\mathbf{e}_{-1})\|.$$

Reminder:

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

It is sufficient to show that

$$\|T(\mathbf{e}_{-1})\|_{\chi} \geq \|T(\mathbf{e}_{-1})\|.$$

For any $\varepsilon > 0$, there exists $q \in H^p(\mathbb{T})$, such that $\|q\|_{H^p} = 1$ and $\|T(\mathbf{e}_{-1})q\|_{H^p} \geq \|T(\mathbf{e}_{-1})\| - \varepsilon$.

Reminder:

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

It is sufficient to show that

$$\|T(\mathbf{e}_{-1})\|_{\chi} \geq \|T(\mathbf{e}_{-1})\|.$$

For any $\varepsilon > 0$, there exists $q \in H^p(\mathbb{T})$, such that $\|q\|_{H^p} = 1$ and $\|T(\mathbf{e}_{-1})q\|_{H^p} \geq \|T(\mathbf{e}_{-1})\| - \varepsilon$.
Take any finite set $\{\varphi_1, \dots, \varphi_m\} \subset H^p(\mathbb{T})$.

Reminder:

Theorem 1

$$\|T(\mathbf{e}_{-1})\|_{\mathrm{e}} = \|T(\mathbf{e}_{-1})\|, \quad \forall p \in (1, \infty).$$

It is sufficient to show that

$$\|T(\mathbf{e}_{-1})\|_{\chi} \geq \|T(\mathbf{e}_{-1})\|.$$

For any $\varepsilon > 0$, there exists $q \in H^p(\mathbb{T})$, such that $\|q\|_{H^p} = 1$ and $\|T(\mathbf{e}_{-1})q\|_{H^p} \geq \|T(\mathbf{e}_{-1})\| - \varepsilon$.
Take any finite set $\{\varphi_1, \dots, \varphi_m\} \subset H^p(\mathbb{T})$. If $N \in \mathbb{N}$ is suffuciently large, then

$$\|T(\mathbf{e}_{-1})(q \circ \mathbf{e}_N) - \varphi_j\|_{H^p} \geq \|T(\mathbf{e}_{-1})\| - 2\varepsilon, \quad j = 1, \dots, m.$$

Open problems and related results

What is the norm of the backward shift operator

$$T(\mathbf{e}_{-1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T}),\ 1 \leq p \leq \infty?$$

Open problems and related results

What is the norm of the backward shift operator

$$T(\mathbf{e}_{-1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T}),\ 1 \leq p \leq \infty?$$

Reminder: $\mathbf{K}_0 f = f(0)$.

$$\|T(\mathbf{e}_{-1})\| = \|\mathrm{I} - \mathbf{K}_0\|_{H^p \to H^p} \leq \|\mathrm{I} - \mathbf{K}_0\|_{L^p \to L^p} \leq 2^{\left|1-\frac{2}{p}\right|} \leq 2.$$

Open problems and related results

What is the norm of the backward shift operator

$$T(\mathbf{e}_{-1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T}),\ 1 \le p \le \infty?$$

Reminder: $\mathbf{K}_0 f = f(0)$.

$$\|T(\mathbf{e}_{-1})\| = \|\mathrm{I} - \mathbf{K}_0\|_{H^p \to H^p} \le \|\mathrm{I} - \mathbf{K}_0\|_{L^p \to L^p} \le 2^{\left|1-\frac{2}{p}\right|} \le 2.$$

$$\|T(\mathbf{e}_{-1})\|_{H^\infty \to H^\infty} = 2 \ \text{ and } \ \|T(\mathbf{e}_{-1})\|_{H^p \to H^p} \to 2 \ \text{ as } \ p \to \infty.$$

Open problems and related results

What is the norm of the backward shift operator

$$T(\mathbf{e}_{-1}) : H^p(\mathbb{T}) \to H^p(\mathbb{T}),\ 1 \le p \le \infty?$$

Reminder: $\mathbf{K}_0 f = f(0)$.

$$\|T(\mathbf{e}_{-1})\| = \|\mathrm{I} - \mathbf{K}_0\|_{H^p \to H^p} \le \|\mathrm{I} - \mathbf{K}_0\|_{L^p \to L^p} \le 2^{\left|1-\frac{2}{p}\right|} \le 2.$$

$\|T(\mathbf{e}_{-1})\|_{H^\infty \to H^\infty} = 2$ and $\|T(\mathbf{e}_{-1})\|_{H^p \to H^p} \to 2$ as $p \to \infty$.

T. Ferguson (arXiv, 2017): $\|T(\mathbf{e}_{-1})\|_{H^1 \to H^1} < 1.7047$.
It follows from the proof that $\|T(\mathbf{e}_{-1})\|_{H^p \to H^p} < 1.7047$ if p is sufficiently close to 1.

T.F. Móri, *Sharp inequalities between centered moments*, 2009
+ complexification (not entirely trivial)

$$\|\mathrm{I}-\mathbf{K}_0\|_{L^p\to L^p}=c_p,\quad 1\le p\le\infty,$$

where

$$c_p:=\max_{0<\alpha<1}\left(\alpha^{p-1}+(1-\alpha)^{p-1}\right)^{\frac{1}{p}}\left(\alpha^{\frac{1}{p-1}}+(1-\alpha)^{\frac{1}{p-1}}\right)^{1-\frac{1}{p}},\qquad 1<p<\infty,$$

$$c_1:=\lim_{p\to 1+0}c_p=2,\quad c_\infty:=\lim_{p\to\infty}c_p=2.$$

T.F. Móri, *Sharp inequalities between centered moments*, 2009
+ complexification (not entirely trivial)

$$\|\mathrm{I} - \mathbf{K}_0\|_{L^p \to L^p} = c_p, \quad 1 \le p \le \infty,$$

where

$$c_p := \max_{0<\alpha<1} \left(\alpha^{p-1} + (1-\alpha)^{p-1}\right)^{\frac{1}{p}} \left(\alpha^{\frac{1}{p-1}} + (1-\alpha)^{\frac{1}{p-1}}\right)^{1-\frac{1}{p}}, \quad 1 < p < \infty,$$

$$c_1 := \lim_{p \to 1+0} c_p = 2, \quad c_\infty := \lim_{p\to\infty} c_p = 2.$$

$c_2 = 1$, $c_{p'} = c_p$ for $p' = \frac{p}{p-1}$, and

$$1 \le c_p \le 2^{\left|1-\frac{2}{p}\right|},$$

where the left inequality is strict unless $p = 2$, while the right one is strict unless $p = 1, 2$ or ∞. Further,

$$c_p \ge 2^{1-\frac{2}{p}} \left(\frac{2}{ep}\right)^{\frac{1}{2p}}, \quad \forall p > 2.$$

Open problems and related results

What are the values of

$$\|\mathrm{I} - \mathbf{K}_n\|_{H^p \to H^p}, \quad \|\mathrm{I} - \mathbf{K}_n\|_{L^p \to L^p},$$
$$M(H^p), \ M^*(H^p), \quad 1 < p < \infty?$$

Open problems and related results

What are the values of

$$\|\mathrm{I}-\mathbf{K}_n\|_{H^p\to H^p},\quad \|\mathrm{I}-\mathbf{K}_n\|_{L^p\to L^p},$$
$$M(H^p),\ M^*(H^p),\quad 1<p<\infty?$$

$$\|\mathrm{I}-\mathbf{K}_n\|_{H^p\to H^p}\geq\|\mathrm{I}-\mathbf{K}_0\|_{H^p\to H^p},$$
$$\|\mathrm{I}-\mathbf{K}_n\|_{L^p\to L^p}\geq\|\mathrm{I}-\mathbf{K}_0\|_{L^p\to L^p},\quad\forall n\in\mathbb{N}.$$

Theorem

$$M(L^p) = M^*(L^p) = c_p, \quad 1 < p < \infty.$$

Theorem

$$M(L^p) = M^*(L^p) = c_p, \quad 1 < p < \infty.$$

T. Oikhberg (2011): Let X be a separable rearrangement invariant non-atomic Banach function space not isometric to L^2. Then $M(X) > 1$.

Theorem

Let $1 \leq p < \infty$ and $T \in \mathcal{K}(L^p)$ be such that $\mathrm{I} - T$ is not invertible. Then

$$\|\mathrm{I} - T\|_{L^p \to L^p} \geq c_p.$$

Theorem

Let $1 \leq p < \infty$ and $T \in \mathcal{K}(L^p)$ be such that $\mathrm{I} - T$ is not invertible. Then

$$\|\mathrm{I} - T\|_{L^p \to L^p} \geq c_p.$$

Corollary

Let $1 \leq p < \infty$ and let $Q : L^p \to L^p$, $Q \neq \mathrm{I}$ be a projection onto a finite-codimensional subspace. Then

$$\|Q\|_{L^p \to L^p} \geq c_p.$$

Theorem

Let $1 \leq p < \infty$ and $T \in \mathcal{K}(L^p)$ be such that $\mathrm{I} - T$ is not invertible. Then

$$\|\mathrm{I} - T\|_{L^p \to L^p} \geq c_p.$$

Corollary

Let $1 \leq p < \infty$ and let $Q : L^p \to L^p$, $Q \neq \mathrm{I}$ be a projection onto a finite-codimensional subspace. Then

$$\|Q\|_{L^p \to L^p} \geq c_p.$$

B. Randrianantoanina (1995): Let X be a separable rearrangement invariant non-atomic Banach function space not isometric to L^2. Then $\|Q\| > 1$ for every projection $Q : X \to X$, $Q \neq \mathrm{I}$ onto a finite-codimensional subspace of X.

$H^p(\mathbb{T})$ is isomorphic (R. P. Boas, Jr, 1955) but not isometric (for $p \neq 2$) to $L^p(\mathbb{T})$.

$H^p(\mathbb{T})$ is isomorphic (R. P. Boas, Jr, 1955)
but not isometric (for $p \neq 2$) to $L^p(\mathbb{T})$.

F. Lancien, B. Randrianantoanina, and E. Ricard (2005): Let $1 \leq p < \infty$, $p \neq 2$ and let $Q : H^p \to H^p$ be a projection onto a subspaces of finite dimension larger than one. Then $\|Q\| > 1$.

$H^p(\mathbb{T})$ is isomorphic (R. P. Boas, Jr, 1955)
but not isometric (for $p \neq 2$) to $L^p(\mathbb{T})$.

F. Lancien, B. Randrianantoanina, and E. Ricard (2005): Let $1 \leq p < \infty$, $p \neq 2$ and let $Q : H^p \to H^p$ be a projection onto a subspaces of finite dimension larger than one. Then $\|Q\| > 1$.

Leo Tolstoy, *Anna Karenina*
“All Hilbert spaces are alike; each Banach space is unhappy in its own way."

$H^p(\mathbb{T})$ is isomorphic (R. P. Boas, Jr, 1955)
but not isometric (for $p \neq 2$) to $L^p(\mathbb{T})$.

F. Lancien, B. Randrianantoanina, and E. Ricard (2005): Let $1 \leq p < \infty$, $p \neq 2$ and let $Q : H^p \to H^p$ be a projection onto a subspaces of finite dimension larger than one. Then $\|Q\| > 1$.

Leo Tolstoy, *Anna Karenina* ("corrected" translation)
“All Hilbert spaces are alike; each Banach space is unhappy in its own way."