# Finite rank perturbations, Clark's model and matrix weights

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1. Main objects: Finite rank perturbations and models
   - Finite rank perturbations
   - Functional models
   - Defects and characteristic function for $T_\Gamma$
2. Toward a formula for the adjoint Clark operator
   - Spectral representation of unitary perturbations
   - Model, agreement of parametrizing operators
   - Representation formula, rank 1 case
3. Clark operator and its adjoint in matrix case
   - A universal formula for adjoint operator in matrix case
   - Adjoint Clark operator $\Phi_\Gamma^*$
   - Direct Clark operator $\Phi_\Gamma$

# Finite rank perturbations

- $U$ unitary in $\mathcal{H}$, subset $R \subset \mathcal{H}$ is fixed, $\dim R = d$.
- Operators $K$, $\operatorname{Ran} K \subset R$, such that $U + K$ is unitary (contraction) are parametrized by $d \times d$ unitary (contractive) matrices $\Gamma$: namely fix unitary $\mathbf{B} : \mathbb{C}^d \to R$ then unitary (contractive) $d \times d$ matrices $\Gamma$ parametrize all unitary (contractive) perturbed operators

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U$$

Indeed, trivial when $U = \mathbf{I}$, and right multiplying by $U$ get the formula.

- Familiar parametrization for rank one perturbations

$$T_\gamma = U + (\gamma - 1)bb^* U = U + (\gamma - 1)b(b_*)^*, \qquad b_* = U^* b.$$

$\|b\| = 1$.

# Finite rank perturbations

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U, \qquad \Gamma : \mathbb{C}^d \to \mathbb{C}^d.$$

- WLOG $R = \operatorname{Ran} \mathbf{B}$ is *-cyclic.
- If $\Gamma$ is a strict contraction, i.e. $\|\Gamma x\| < \|x\|\ \forall x$, then $T_\Gamma$ is a completely non-unitary (c.n.u.) contraction.
  C.n.u. means that there is no a reducing subspace on which the operator is unitary.
- As c.n.u. $T_\Gamma$ admits a functional model $\mathcal{M}_\Gamma = \mathcal{M}_{T_\Gamma}$

# Goal

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U, \qquad \Gamma : \mathbb{C}^d \to \mathbb{C}^d,\ \|\Gamma\| < 1.$$

- Consider $U$ in its spectral representation.
- We assumed that $\operatorname{Ran}\mathbf{B}$ is *-cyclic, so $T_\Gamma$ is c.n.u.
- $T_\Gamma$ is unitarily equivalent to its functional model $\mathcal{M}_\Gamma : \mathcal{K}_\theta \to \mathcal{K}_\theta$, (for example Sz.-Nagy-Foiaș model), where $\theta = \theta_T$ is the characteristic function.
- Want to describe the Clark operator, i.e. a unitary operator $\Phi = \Phi_\Gamma$ such that

$$T_\Gamma \Phi_\Gamma = \Phi_\Gamma \mathcal{M}_\Gamma.$$

# $U$ in spectral representation

- WLOG assume that $U = M_\xi$ in

$$\mathcal{H} = \int_{\mathbb{T}}^{\oplus} E(\xi) \mathrm{d}\mu(\xi),$$

$E(\xi) = \operatorname{span}\{e_k : 1 \le k \le N(\xi)\} \subset E$, $\{e_k\}_k$— ONB in $E$.

- $\mathcal{H} \subset L^2(\mu; E)$:

$$\mathcal{H} = \{f \in L^2(\mu; E) : f(\xi) \in E(\xi) \ \mu\text{-a.e.}\}.$$

- Define matrix function $B$, $B(\xi) : \mathbb{C}^d \to E(\xi) \subset E$,

$$B(\xi)e = \mathbf{B}e(\xi), \qquad e \in \mathbb{C}^d.$$

- $\operatorname{Ran} \mathbf{B}$ is *-cyclic iff

$$\operatorname{Ran} B(\xi) = E(\xi) \qquad \mu\text{-a.e.}$$

# Functional model for a a c.n.u. contraction.

- The model $\mathcal{M}$ for a contraction is not a multiplication operator, it cannot be.
- It is a *compression* of a multiplication operator

$$\mathcal{M} = P_{\mathcal{K}} M_z \Big|_{\mathcal{K}},$$

where $\mathcal{K}$ is an appropriate subspace of a (generally vector valued) $L^2$ space.

- The vector-valued $L^2$ space comes from the spectral representation of the minimal unitary dilation $U$ of $T$ (will be explained later)
- The *characteristic function* $\theta$ is a unitary invariant of $T$ and main object in the theory of the model.

Following Nikolskii–Vasyunin [7] the functional model is constructed as follows:

1. For a contraction $T : \mathcal{K} \to \mathcal{K}$ consider its *minimal* unitary dilations $\mathcal{U} : \mathcal{H} \to \mathcal{H}$, $\mathcal{K} \subset \mathcal{H}$,
$$T^n = P_{\mathcal{K}} \mathcal{U}^n \,\Big|\, \mathcal{K}, \qquad n \geq 0.$$
2. Pick a spectral representation of $\mathcal{U}$
3. Work out formulas in this spectral representation
4. Model subspace $\mathcal{K} = \mathcal{K}_\theta$ is usually a subspace of a weighted space $L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W)$, $\mathfrak{D} \cong \mathfrak{D}_T$, $\mathfrak{D}_* \cong \mathfrak{D}_{T^*}$ with some operator-valued weight.
5. Model operator $\mathcal{M}$ is a compression of the model for $\mathcal{U}$, i.e. of the multiplication operator, $\mathcal{M} = P_{\mathcal{K}} M_z \big|_{\mathcal{K}}$.

Specific representations give us a *transcription* of the model.
Among common transcriptions are: the Sz.-Nagy–Foiaș transcription, the de Branges–Rovnyak transcription, Pavlov transcription.

# Characteristic function

Let $T$ be a c.n.u.
Defect operators and subspaces,

$$D_T := (\mathbf{I} - T^*T)^{1/2}, \qquad D_{T^*} := (\mathbf{I} - TT^*)^{1/2},$$
$$\mathfrak{D}_T := \operatorname{clos}\operatorname{Ran} D_T, \qquad \mathfrak{D}_{T^*} := \operatorname{clos}\operatorname{Ran} D_{T^*}.$$

Let $\dim\mathfrak{D} = \dim\mathfrak{D}_T$, $\dim\mathfrak{D}_* = \dim\mathfrak{D}_{T^*}$, and let

$$V : \mathfrak{D}_T \to \mathfrak{D}, \qquad V_* : \mathfrak{D}_{T^*} \to \mathfrak{D}_*$$

be unitary operators (coordinate operators).
The characteristic function $\theta = \theta_T = \theta_{T,V,V_*}$, $\theta(z) : \mathfrak{D} \to \mathfrak{D}_*$ is defined as

$$\theta_T(z) = V_* \left( -T + zD_{T^*} \left(\mathbf{I}_{\mathcal{H}} - zT^*\right)^{-1} D_T \right) V^* \Big|_{\mathfrak{D}}, \qquad z \in \mathbb{D}.$$

# Sz.-Nagy–Foiaș and de Branges–Rovnyak transcriptions

- **Sz.-Nagy–Foiaș:** $\mathcal{H} = L^2(\mathfrak{D}_* \oplus \mathfrak{D})$ (non-weighted, $W \equiv I$).

$$\mathcal{K}_\theta := \begin{pmatrix} H^2_{\mathfrak{D}_*} \\ \operatorname{clos} \Delta L^2_{\mathfrak{D}} \end{pmatrix} \ominus \begin{pmatrix} \theta \\ \Delta \end{pmatrix} H^2_{\mathfrak{D}},$$

where $\Delta(z) := (1 - \theta(z)^*\theta(z))^{1/2}$, $z \in \mathbb{T}$.

- **de Branges–Rovnyak:** $\mathcal{H} = L^2(\mathfrak{D}_* \oplus \mathfrak{D}, W_\theta^{[-1]})$, where

$$W_\theta(z) = \begin{pmatrix} I & \theta(z) \\ \theta(z)^* & I \end{pmatrix}$$

and $W_\theta^{[-1]}$ is the Moore–Penrose inverse of $W_\theta$. $\mathcal{K}_\theta$ is given by

$$\left\{ \begin{pmatrix} g_+ \\ g_- \end{pmatrix} : \; g_+ \in H^2(\mathfrak{D}_*),\; g_- \in H^2_-(\mathfrak{D}),\; g_- - \theta^* g_+ \in \Delta L^2(\mathfrak{D}) \right\}.$$

# Defects and characteristic function for $T_\Gamma$

Recall: $T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U, \qquad \Gamma : \mathbb{C}^d \to \mathbb{C}^d, \quad \|\Gamma\| < 1.$

- $\mathfrak{D}_{T_\Gamma} = \operatorname{Ran}(\mathbf{B}^* U)^* = \operatorname{Ran} U^* \mathbf{B}$ and $\mathfrak{D}_{T_\Gamma^*} = \operatorname{Ran} \mathbf{B}$
- In the scalar case $\mathfrak{D}_{T_\gamma}$ and $\mathfrak{D}_{T_\gamma^*}$ are spanned by the vectors $\bar\xi$ and $\mathbf{1}$ respectively.
- Characteristic function $\theta_T$ of a contraction $T$ is defined as

$$\theta_T(z) = V_* \left( -T + z D_{T^*} \left( \mathbf{I}_\mathcal{H} - z T^* \right)^{-1} D_T \right) V^* \Big|_{\mathfrak{D}}, \qquad z \in \mathbb{D}.$$

In our case $V_* = \mathbf{B}^*$, $V = (\mathbf{B}^* U)^* = U^* \mathbf{B}$,

$$T_\Gamma = U + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U, \qquad \Gamma : \mathbb{C}^d \to \mathbb{C}^d, \quad \|\Gamma\| < 1.$$

and $(\mathbf{I} - zU^*)^{-1}$ is just the multiplication by $(1 - z\bar\xi)^{-1}$.

- To compute it use Woodbury inversion formula:
  if $B, C : E \to \mathcal{H}$ (in applications $\dim E$ is small), then

$$(\mathbf{I}_{\mathcal{H}} - CB^*)^{-1} = \mathbf{I}_{\mathcal{H}} + C(\mathbf{I}_E - B^*C)^{-1}B^*.$$

  To get this formula just decompose $(\mathbf{I}_{\mathcal{H}} - CB^*)^{-1}$ using geometric series. A formal proof can be obtained just by checking.
- In rank one case we get the Sherman–Morrison inversion formula:

$$(I - cb^*)^{-1} = I + \frac{1}{d}cb^*, \qquad d = (c, b) = b^*c.$$

- $I - zT_\Gamma^*$ is a finite rank perturbation of $I - zU_1^* = I - zM_{\overline{\xi}}$;
- The inverse of $I - zM_{\overline{\xi}}$ is multiplication by $(1 - z\overline{\xi})^{-1}$, so Cauchy integrals appear.

# Cauchy Transforms

- Define Cauchy integrals

$$\mathcal{C}_1\tau(z) := \int_{\mathbb{T}} \frac{\overline{\xi} z d\tau(\xi)}{1-\overline{\xi} z}, \qquad \mathcal{C}_2\tau(z) := \int_{\mathbb{T}} \frac{1+\overline{\xi} z}{1-\overline{\xi} z} d\tau(\xi).$$

- Consider matrix-valued measure $B(\xi)^* B(\xi) \mathrm{d}\mu(\xi)$ ($B^* B\mu$ as shorthand), and let

$$F_1(z) := \mathcal{C}_1[B^* B\mu](z), \qquad F_2(z) := \mathcal{C}_2[B^* B\mu](z), \qquad z \in \mathbb{D}$$

be the corresponding matrix-valued Cauchy transforms

# Characteristic function for $T_\Gamma$

- Characteristic function $\theta_\gamma$ of $T_\gamma$:

$$\theta_\gamma(z) = -\gamma + \frac{(1-|\gamma|^2)\mathcal{C}_1\mu(z)}{1+(1-\overline{\gamma})\mathcal{C}_1\mu(z)} = \frac{(1-\gamma)\mathcal{C}_2\mu(z)-(1+\gamma)}{(1-\overline{\gamma})\mathcal{C}_2\mu(z)+(1+\overline{\gamma})},$$

- Note that $\theta_\gamma(0) = -\gamma$, because $\mathcal{C}_1\mu(0) = 0$
- In the matrix case

$$\begin{aligned}\theta_\Gamma(z) &= -\Gamma + D_{\Gamma^*}F_1(z)\Big(\mathbf{I}_{\mathfrak{D}} - (\Gamma^* - \mathbf{I}_{\mathfrak{D}})F_1(z)\Big)^{-1}D_\Gamma \\ &= -\Gamma + D_{\Gamma^*}\Big(\mathbf{I}_{\mathfrak{D}} - F_1(z)(\Gamma^* - \mathbf{I}_{\mathfrak{D}})\Big)^{-1}F_1(z)D_\Gamma,\end{aligned}$$

# Characteristic function for $T_{\mathbf{0}}$

- For $\gamma = 0$

$$\theta_0(z) = \frac{\mathcal{C}_1\mu(z)}{1+\mathcal{C}_1\mu(z)} = \frac{\mathcal{C}_2\mu(z)-1}{\mathcal{C}_2\mu(z)+1}, \qquad z \in \mathbb{D}.$$

- For $\Gamma = \mathbf{0}$

$$\begin{aligned}\theta_{\mathbf{0}}(z) &= F_1(z)(\mathbf{I}+F_1(z))^{-1} = (\mathbf{I}+F_1(z))^{-1}F_1(z)\\ &= (F_2(z)-\mathbf{I})(F_2(z)+\mathbf{I})^{-1} = (F_2(z)+\mathbf{I})^{-1}(F_2(z)-\mathbf{I}).\end{aligned}$$

# LFTs for characteristic functions

- In the scalar case

$$\theta_\gamma(z) = \frac{\theta_0(z) - \gamma}{1 - \overline{\gamma}\theta_0(z)} ,$$

- In the matrix case

$$\begin{aligned} \theta_\Gamma &= D_{\Gamma^*}^{-1}(\theta_{\mathbf{0}} - \Gamma)(\mathbf{I}_{\mathfrak{D}} - \Gamma^*\theta_{\mathbf{0}})^{-1} D_\Gamma \\ &= D_{\Gamma^*}(\mathbf{I}_{\mathfrak{D}} - \theta_{\mathbf{0}}\Gamma^*)^{-1}(\theta_{\mathbf{0}} - \Gamma) D_\Gamma^{-1} \end{aligned}$$

# "Model" case of rank one unitary perturbations

Recall: $U_\alpha = U_1 + (\alpha - 1)b(b_*)^*$, $|\alpha| = 1$

$$U_1 = M_\xi \text{ in } L^2(\mu), \quad \mu(\mathbb{T}) = 1, \qquad b \equiv 1, \quad b_* = U_1^* b \equiv \overline{\xi}$$

- Let $\mu_\alpha$ be the spectral measure of $U_\alpha$ corresponding to the vector $b$.
- Want to find a unitary operator $\mathcal{V}_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ such that $\mathcal{V}_\alpha b = \mathbf{1} \in L^2(\mu_\alpha)$ and such that

$$\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha .$$

Case of self-adjoint perturbations was treated earlier by Liaw–Treil in [3]. This case is treated similarly.

# Pretending to be a physicist

Let $\mathcal{V}_\alpha$ be an integral operator with kernel $K(z,\xi)$.

- $U_\alpha = M_\xi + (\alpha - 1)bb_*^*$, so we can rewrite the relation $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha M_\xi = M_z \mathcal{V}_\alpha - (1-\alpha)\mathcal{V}_\alpha bb_*^*.$$

- We know that $\mathcal{V}_\alpha b = 1$, $b_* = \overline{\xi}$, so $\mathcal{V}_\alpha bb_*^*$ is an integral operator with kernel $\xi$

$$K(z,\xi)\xi = zK(z,\xi) - (\alpha - 1)\xi.$$

- Solving for $K$ we get

$$K(z,\xi) = (1-\alpha)\frac{\xi}{\xi - z} = (1-\alpha)\frac{1}{1-\overline{\xi}z}$$

# Commutation relations and Cauchy type integrals

## A general principle

Rank one commutation relations like

$$\mathcal{V} M_\xi = M_z \mathcal{V} + c b^*$$

usually give singular integral representations for $\mathcal{V}$.

# First representation for $\mathcal{V}_\alpha$

**Theorem (Repesentation of $\mathcal{V}_\alpha$)**

*The unitary operator* $\mathcal{V}_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ *such that* $\mathcal{V}_\alpha b = \mathbf{1} \in L^2(\mu_\alpha)$ *and such that*

$$\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha .$$

*is given by*

$$\mathcal{V}_\alpha f(z) = f(z) + (1-\alpha) \int_{\mathbb{T}} \frac{f(\xi) - f(z)}{1 - \bar{\xi} z} \, d\mu(\xi)$$

*for* $f \in C^1(\mathbb{T})$

# Idea of the proof

- Recalling that $U_\alpha = U_1 + (\alpha - 1) b b_*^*$ rewrite $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as

$$\mathcal{V}_\alpha U_1 = M_z \mathcal{V}_\alpha + (1-\alpha)(\mathcal{V}_\alpha b) b_*^*$$

# Idea of the proof

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$$\mathcal{V}_\alpha U_1 = M_z \mathcal{V}_\alpha + (1-\alpha)(\mathcal{V}_\alpha b)b_*^*$$

- Right multiplying by $U_1$ we get

$$\mathcal{V}_\alpha U_1 U_1 = M_z \mathcal{V}_\alpha U_1 + (1-\alpha)(\mathcal{V}_\alpha b)b_*^* U_1.$$

and applying the previous identity to $\mathcal{V}_\alpha U_1$ in the right hand side, we get

$$\mathcal{V}_\alpha U_1^2 = M_z^2 \mathcal{V}_\alpha + (1-\alpha)\left[(M_z \mathcal{V}_\alpha b)b_*^* + (\mathcal{V}_\alpha b)b_*^* U_1\right]$$

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- By induction we get
$$\mathcal{V}_\alpha U_1^n = M_z^n \mathcal{V}_\alpha + (1-\alpha)\sum_{k=1}^{n} M_z^{k-1}(\mathcal{V}_\alpha b)b_*^* U_1^{n-k}.$$

# Idea of the proof

- Recalling that $U_\alpha = U_1 + (\alpha - 1)bb_*^*$ rewrite $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$ as
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and applying the previous identity to $\mathcal{V}_\alpha U_1$ in the right hand side, we get
$$\mathcal{V}_\alpha U_1^2 = M_z^2 \mathcal{V}_\alpha + (1-\alpha)\left[(M_z \mathcal{V}_\alpha b)b_*^* + (\mathcal{V}_\alpha b)b_*^* U_1\right]$$
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$$\mathcal{V}_\alpha U_1^n = M_z^n \mathcal{V}_\alpha + (1-\alpha)\sum_{k=1}^{n} M_z^{k-1}(\mathcal{V}_\alpha b)b_*^* U_1^{n-k}.$$
- Applying to $b \equiv 1$ and summing geometric progression we get the formula for $f(\xi) = \xi^n$, $n \geq 0$.

# Idea of the proof, continued

- To get the formula for $\overline{\xi}^n$ we use $\mathcal{V}_\alpha U_\alpha^* = M_{\overline{z}} \mathcal{V}_\alpha$, which is obtained by taking adjoint in $\mathcal{V}_\alpha U_\alpha = M_z \mathcal{V}_\alpha$.
- Extend the formula from trig. polynomials to $f \in C^1$ by standard approximation reasoning.

# Idea of the proof, continued

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- Extend the formula from trig. polynomials to $f \in C^1$ by standard approximation reasoning.

## A general statement

Rank one commutation relations like

$$\mathcal{V}M_\xi = M_z\mathcal{V} + cb^*$$

usually give singular integral representations for $\mathcal{V}$.

# Singular integral operators

Recall that $\mathcal{V}_\alpha f(z) = f(z) + (1-\alpha)\int_{\mathbb{T}} \frac{f(\xi)-f(z)}{1-\bar{\xi}z}\,d\mu(\xi)$

## Theorem (Regularization of the weighted Cauchy transform)

*The integral operators $T_r = T_r^\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ with kernels $1/(1-r\bar{\xi}z)$, $r \in \mathbb{R}_+ \setminus \{1\}$ are uniformly bounded.*

- Let $Tf(z) := \int_{\mathbb{T}} \frac{f(\xi)}{1-\bar{\xi}z} d\mu(\xi)$; well defined for $z \notin \operatorname{supp} f$
- Since $\mathcal{V}_\alpha$ is bounded, we get for $f, g \in C^1$, $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$

$$(Tf, g)_{L^2(\mu_\alpha)} \le C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu_\alpha)}$$

- By a theorem of Liaw–Treil [4] this implies uniform boundedness of the regularizations $T_r$ if the measures $\mu$ and $\mu_\alpha$ do not have common atoms ($U_1$ and $U_\alpha$ do not have common eigenvalues).

# Singular integral operators

- Uniform boundedness of $T_r$ together with $\mu_\alpha$-a.e. convergence of $T_r f$ imply existence of w.o.t.-limits $T^\alpha_\pm = \text{w.o.t.-}\lim_{r\to 1\mp} T_r$.
- Using $T^\alpha_\pm$ we can rewrite the representation

$$\mathcal{V}_\alpha f(z) = f(z) + (1-\alpha)\int_{\mathbb{T}} \frac{f(\xi)-f(z)}{1-\bar{\xi}z}\, d\mu(\xi)$$

  as

$$\mathcal{V}_\alpha f = [\mathbf{1} - (1-\alpha)T^\alpha_\pm \mathbf{1}]f + (1-\alpha)T^\alpha_\pm f.$$

- $T^\alpha_\pm : L^2(\mu) \to L^2(\mu_\alpha)$, $T^\alpha_\pm f$ is given by boundary values of $\mathcal{C}[f\mu]$, $\mathcal{C}\tau(z) = \int_{\mathbb{T}}(1-\bar{\xi}z)^{-1}\mathrm{d}\tau(\xi)$.
- $(\mu_\alpha)_{\mathrm{a}}$-a.e. convergence follows from classical results about jumps of Cauchy transform; $(\mu_\alpha)_{\mathrm{s}}$-a.e. convergence can be obtained from Poltoratskii's theorem about boundary values of the normalized Cauchy transform, see [10].
- For the weak convergence it is enough to have $\mu_\alpha$-a.e. convergence of $T_r f$ for $f \in C^1$, which can be proved using elementary methods.

# Model, agreement of coordinate and parametrizing operators

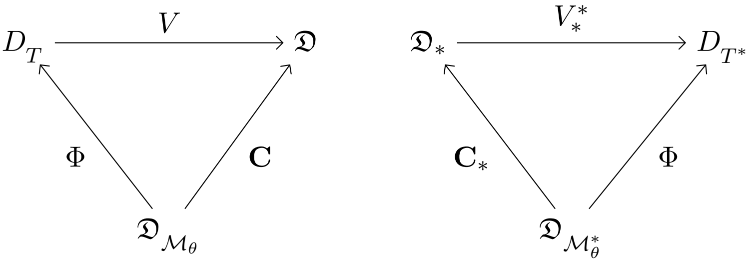
- Let $T$ be a c.n.u. contraction, $V : \mathfrak{D}_T = \mathfrak{D}$, $V : \mathfrak{D}_{T^*} = \mathfrak{D}_*$ unitary operators (coordinate operators),
- $\theta = \theta_{T,V,V_*} \in H^\infty(\mathfrak{D} \to \mathfrak{D}_*)$ its characteristic function, $\mathcal{M}_\theta : \mathcal{K}_\theta \to \mathcal{K}_\theta$ the model operator.
- We say that unitary $\mathbf{C} : \mathfrak{D} \to \mathfrak{D}_{\mathcal{M}_\theta}$, $\mathbf{C}_* : \mathfrak{D}_* \to \mathfrak{D}_{\mathcal{M}_\theta^*}$ agree with $V$, $V_*$ if

$$\mathbf{C}^* = V\Phi \Big|_{\mathfrak{D}_{\mathcal{M}_\theta}}, \qquad \mathbf{C}_*^* = V_*\Phi \Big|_{\mathfrak{D}_{\mathcal{M}_\theta^*}} .$$

for a unitary $\Phi : \mathcal{K}_\theta \to \mathcal{H}$ such that $T\Phi = \Phi\mathcal{M}_\theta$

# Model, agreement of coordinate and parametrizing operators

In other words, the following diagrams commute:



# Model: agreement

In the Sz.-Nagy–Foiaș notation

$$\mathbf{C}_* e_* = \begin{pmatrix} \mathbf{I} - \theta(z)\theta^*(0) \\ -\Delta(z)\theta^*(0) \end{pmatrix} \left(\mathbf{I} - \theta(0)\theta^*(0)\right)^{-1/2} e_*, \qquad e_* \in \mathfrak{D}_*,$$

$$\mathbf{C} e = \begin{pmatrix} z^{-1}\left(\theta(z) - \theta(0)\right) \\ z^{-1}\Delta(z) \end{pmatrix} \left(\mathbf{I} - \theta^*(0)\theta(0)\right)^{-1/2} e, \qquad e \in \mathfrak{D},$$

For the Clark case $T = T_\Gamma = T + \mathbf{B}(\Gamma - \mathbf{I})\mathbf{B}^* U$, $V = U^*\mathbf{B}$, $V_* = \mathbf{B}$, $\mathfrak{D} = \mathfrak{D}_* = \mathbb{C}^d$ we get, noticing that $\theta(0) = -\Gamma$ that

$$\mathbf{C}e(z) = C(z)e, \qquad \mathbf{C}_* e(z) = C_*(z)e,$$

where

$$C_*(z) = \begin{pmatrix} \mathbf{I} + \theta(z)\Gamma^* \\ \Delta(z)\Gamma^* \end{pmatrix} D_{\Gamma^*}^{-1},$$

$$C(z) = z^{-1}\begin{pmatrix} \theta(z) + \Gamma \\ \Delta(z) \end{pmatrix} D_{\Gamma}^{-1};$$

## Theorem (A "universal" representation formula)

*In the rank one case the adjoint Clark operator $\Phi^*$, ($C$, $C_*$ agree with Clark model) is given for $f \in C^1(\mathbb{T})$ by*

$$\Phi_\gamma^* f(z) = C_*(z) f(z) + C_1(z) \int \frac{f(\xi) - f(z)}{1 - \overline{\xi} z}\, d\mu(\xi), \quad z \in \mathbb{T},$$

*where $C_1(z) = C_*(z) - zC(z)$*

Regularizing Cauchy Transform we get the following representation of the $\Phi^*$,

$$\Phi^* f(z) = A(z) f(z) + C_1(z) \mathcal{C}_+[f\mu](z),$$

where $A = C_* - C_1 \mathcal{C}_+ \mu$,

$$\mathcal{C}\tau(z) = \int_{\mathbb{T}} \frac{1}{1 - \overline{\xi} z} \mathrm{d}\tau(\xi).$$

$\mathcal{C}_+$ means boundary values of $\mathcal{C}\tau(z)$, $z \in \mathbb{D}$.

# Idea of the proof

- Write, denoting $C_2(z) := zC(z)$,

$$\begin{aligned}\mathcal{M}_{\theta_\gamma} &= M_z - C_2 C^* - \theta_\gamma(0) C_* C^* \\ &= M_z + (\gamma C_* - C_2) C^*.\end{aligned}$$

Rank one perturbation of $M_z$! Should get at most rank 2 commutation relation.

# Idea of the proof

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$$\begin{aligned} \mathcal{M}_{\theta_\gamma} &= M_z - C_2C^* - \theta_\gamma(0)C_*C^* \\ &= M_z + (\gamma C_* - C_2)C^*. \end{aligned}$$

  Rank one perturbation of $M_z$! Should get at most rank 2 commutation relation.

- Using this identity rewrite $\Phi_\gamma^* T_\gamma = \mathcal{M}_{\theta_\gamma}\Phi_\gamma^*$ as

$$\Phi_\gamma^* U + (\gamma - 1)C_* b^* U = M_z \Phi_\gamma^* + (\gamma C_* - C_2)b^* U$$

  or equivalently

$$\Phi_\gamma^* U = M_z \Phi_\gamma^* + (C_* - C_2)b^* U.$$

  We got rank one commutation relation!

- Commutation relations imply integral representation.

# Idea of the proof, difficulties

- Formally the right side of

$$\Phi_\gamma^* U = M_z \Phi_\gamma^* + (C_* - C_2) b^* U. \qquad (*)$$

  acts from $L^2(\mu)$ to outside of $\mathcal{K}_\theta$.
- To get $\Phi_\gamma^* \overline{\xi}^n$ we use the commutant relation

$$\begin{aligned} \Phi_\gamma^* U^* &= M_{\overline{z}} \Phi_\gamma^* + (C - M_{\overline{z}} C_*) b^* \\ &= M_{\overline{z}} \Phi_\gamma^* - M_{\overline{z}} (C_* - C_2) b^*, \end{aligned}$$

  which cannot be obtained by taking the adjoint of $(*)$.
- It is a miracle that the formulas for $\Phi_\gamma^* \xi^n$ and $\Phi_\gamma^* \overline{\xi}^n$ agree.

**Universal formula:** for $b \in \operatorname{Ran} \mathbf{B}$ and scalar $h \in C^1(\mathbb{T})$

$$(\Phi^* hb)(z) = h(z)C_*(z)\mathbf{B}^* b + C_1(z) \int_{\mathbb{T}} \frac{h(\xi) - h(z)}{1 - z\overline{\xi}} B^*(\xi) b(\xi) \mathrm{d}\mu(\xi)$$

where, recall $C_1(z) = C_*(z) - zC(z)$.

- Matrix function $B$ is defined by $B(\xi)e = (\mathbf{B}e)(\xi)$, $e \in \mathbb{C}^d$, so $\mathbf{B}^* b = \int_{\mathbb{T}} B(\xi)^* b(\xi) \mathrm{d}\mu(\xi)$.
- As in the scalar case, $\Phi^*$ has Cauchy transform part, plus multiplication part.
- Cauchy transform part is easy (put $f = hb$),

$$f \mapsto C_1 \mathcal{C}_+[B^* f \mu], \qquad f \in \mathcal{H} \subset L^2(\mu; E).$$

where,recall

$$\mathcal{C}\tau(z) = \int_{\mathbb{T}} \frac{1}{1 - \overline{\xi} z} \mathrm{d}\tau(\xi).$$

and $\mathcal{C}_+$ means boundary values of $\mathcal{C}\tau(z)$, $z \in \mathbb{D}$.

# Representation in the Sz.-Nagy–Foiaș transcription

Denote by $F = \mathcal{C}_+[B^*B\mu]$. Recall $\Delta_\Gamma : (\mathbf{I} - \theta_\Gamma^*\theta_\Gamma)^{1/2}$.

- The adjoint Clark operator $\Phi^* : \mathcal{H} \subset L^2(\mu : E) \to \mathcal{K}_\theta$ is given by

$$\Phi^* f = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} f + \begin{pmatrix} (\mathbf{I} + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* - \mathbf{I}) \end{pmatrix} \mathcal{C}_+[B^* f \mu],$$

  with $\Psi_2(z) = \widetilde{\Psi}_2(z) R(z)$, where

$$\widetilde{\Psi}_2(z) = \Delta_\Gamma D_\Gamma^{-1}(\Gamma^* + (\mathbf{I} - \Gamma^*)F(z))$$
$$= \Delta_\Gamma D_\Gamma^{-1}(\mathbf{I} - \Gamma^*\theta_{\mathbf{0}}(z))F(z) \qquad \text{a.e. on } \mathbb{T},$$

  and $R$ is a measurable right inverse for the matrix-valued function $B$.

- Formula does not depend on the choice of $R$, because $\mu_{\mathrm{ac}}$-a.e.

$$\widetilde{\Psi}_2^* \widetilde{\Psi}_2 = F^* \Delta_{\mathbf{0}}^2 F = B^* B w$$

  and so $\Psi_2(\xi)^* \Psi_2(\xi) = w(\xi) \mathbf{I}_{E(\xi)}$; here $w$ is the density of $\mu$

# Matrix case: spectral representation with matrix weight

Consider the weighted space $L^2(B^*B\mu)$,

$$\|f\|^2_{L^2(B^*B\mu)} := \int_{\mathbb{T}} \big(B(\xi)^* B(\xi) f(\xi), f(\xi)\big)_{\mathbb{C}^d} \mathrm{d}\mu(\xi)$$

- The operator $\mathcal{U} : L^2(B^*B\mu) \to \mathcal{H}$, $\mathcal{U}f = Bf$ is unitary.
- The adjoint Clark operator $\Phi^* : L^2(B^*B\mu) \to \mathcal{K}_\theta$ is given by

$$\Phi^* f = \begin{pmatrix} 0 \\ \widetilde{\Psi}_2 \end{pmatrix} f + \begin{pmatrix} (\mathbf{I} + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta D_\Gamma^{-1} (\Gamma^* - \mathbf{I}) \end{pmatrix} \mathcal{C}_+[B^* B f \mu],$$

where

$$F = \mathcal{C}_+[B^*B\mu], \qquad \widetilde{\Psi}_2(z) = \Delta D_\Gamma^{-1}(\Gamma^* + (\mathbf{I} - \Gamma^*)F(z))$$

# Direct Clark operator (a.c. part)

Let $\Phi^*_\Gamma f = h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. We computed that

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} f + \begin{pmatrix} (\mathbf{I} + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta_\Gamma D_\Gamma^{-1} (\Gamma^* - \mathbf{I}) \end{pmatrix} \mathcal{C}_+[B^* f \mu].$$

Subtract from the second component an appropriate left multiple of the first component to get rid of $\mathcal{C}_+[B^* f \mu]$:

$$\Psi_2 f = h_2 - \Delta_\Gamma D_\Gamma^{-1} (\Gamma^* - \mathbf{I}) F D_{\Gamma^*} (\mathbf{I} + \theta_\Gamma \Gamma^*)^{-1} h_1$$

Left multiplying by $\Psi_2^*$ and using $\Psi_2^* \Psi_2 = w(\xi) \mathbf{I}_{E(\xi)}$, we get a.c. part

$$\begin{aligned} wf &= R^* F^* (\mathbf{I} - \theta_{\mathbf{0}}^* \Gamma) D_\Gamma^{-1} \Delta_\Gamma h_2 \\ &\qquad - R^* F^* (\mathbf{I} - \theta_{\mathbf{0}}^* \Gamma) D_\Gamma^{-1} \Delta_\Gamma^2 D_\Gamma^{-1} (\Gamma^* - \mathbf{I}) F D_{\Gamma^*} (\mathbf{I} + \theta_\Gamma \Gamma^*)^{-1} h_1 \\ &= R^* F^* (\mathbf{I} - \theta_{\mathbf{0}}^* \Gamma) D_\Gamma^{-1} \Delta_\Gamma h_2 \\ &\qquad - R^* F^* \Delta_{\mathbf{0}}^2 (\mathbf{I} - \Gamma^* \theta_{\mathbf{0}})^{-1} (\Gamma^* - \mathbf{I}) F D_{\Gamma^*} (\mathbf{I} + \theta_\Gamma \Gamma^*)^{-1} h_1 \end{aligned}$$

# Direct Clark operator (singular part)

## Lemma (A. Poltoratskii)

*Let $f \in L^2(\mathbb{T}, \mu; \mathbb{C}^d)$. Then the nontangential boundary values of $\mathcal{C}[f\mu](z)/\mathcal{C}[\mu](z)$, $z \in \mathbb{D}$ exist and equal $f(\xi)$, $\mu_s$-a.a. $\xi \in \mathbb{T}$.*

We had

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_2 \end{pmatrix} f + \begin{pmatrix} (\mathbf{I} + \theta_\Gamma \Gamma^*) D_{\Gamma^*}^{-1} F^{-1} \\ \Delta_\Gamma D_\Gamma^{-1} (\Gamma^* - \mathbf{I}) \end{pmatrix} \mathcal{C}_+[B^* f \mu].$$

Divide by $\mathcal{C}[\mu]$ and solve $\mu_s$-a.e. for $B^* f$ in the first component:

$$B^* f = \frac{1}{\mathcal{C}[\mu]} F D_{\Gamma^*} (\mathbf{I} + \theta_\Gamma \Gamma^*)^{-1} h_1 \qquad \mu_s\text{-a.e.}$$

Left multiplying this identity by $R^*$ we get that

$$\Phi h = f = \frac{1}{\mathcal{C}[\mu]} R^* F D_{\Gamma^*} (\mathbf{I} + \theta_\Gamma \Gamma^*)^{-1} h_1 \qquad \mu_s\text{-a.e.}$$

# Comparison with Clark model

- D. Clark started with model operator $\mathcal{M}_\theta$, ($\theta$ inner $\iff$ $\mu$ is purely singular) and considered it all unitary rank one perturbations.
- In our model it corresponds considering operator $U_\gamma = U_1 + (\gamma - 1)bb_1^*$, $\gamma = -\theta(0)$, then all unitary rank one perturbations are exactly the operators $U_\alpha$, $|\alpha| = 1$.
- Clark measures $\widetilde{\mu}_\alpha$ are the spectral measures of the operators $U_\alpha$.
- If $\theta(0) = 0$ them $\widetilde{\mu}_\alpha = \mu_\alpha$ and the Clark operators coincide with ours.
- If $\theta(0) \neq 0$ $\widetilde{\mu}_\alpha$ is a multiple $\mu_\alpha$, and the operators differ by a factor $c(\gamma)$.
- In Clark model $\widetilde{\mu}_\alpha$ is not a probability measure, $|c(\gamma)|$ compensate for that.

# Comparison with Sarason's model

- D. Sarason in [11] presented a unitary operator between $H^2(\mu) = \overline{\text{span}}\{z^n : n \in \mathbb{Z}_+\}$ and the de Branges space $\mathcal{H}(\theta)$; like Clark, he started with a model operator in $\mathcal{K}_\theta$
- The space $\mathcal{H}(\theta) \subset H^2$ is defined as a range $(I - T_\theta T_{\theta^*})^{1/2} H^2$ endowed with the *range norm* (the minimal norm of the preimage); $T_\varphi : H^2 \to H^2$ is a Toeplitz opearator, $T_\varphi f = P_{H^2}(\varphi f)$.
- If $\theta$ is an extreme point of the unit ball in $H^\infty$ $\left(\int_{\mathbb{T}} \ln(1 - |\theta|^2)|dz| = -\infty \iff \int_{\mathbb{T}} \ln w|dz| = -\infty\right.$, $w$ density of $\mu$) then $\mathcal{H}(\theta)$ is canonically isomorphic to the model space $\mathcal{K}_\theta$ in the de Branges–Rovnyak transcription, see [9].
- His measure $\mu$ coincides with the Clark measure $\widetilde{\mu}_\alpha$,

$$\alpha = \frac{1+\gamma}{1+\overline{\gamma}};$$

the formulas are the same as Clark's.

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