Spectral theory of Hankel operators and related topics
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ABSTRACTS (version 07/10/2017)

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Hankel matrices of indeterminate moment problems

Given a sequence $(s_n)_{n\geq 0}$ of real numbers, we consider the infinite Hankel matrix $\mathcal{H} = \{s_{j+k}\}_{j,k=0}^{\infty}$. Hamburger proved around 1920 that $\mathcal{H}$ is positive definite if and only if

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0 \tag{1}$$

for a positive measure μ on $\mathbb{R}$ with infinite support.

The moment sequence (s_n) can be **determinate** or **indeterminate** in the sense that (1) can have exactly one or several solutions μ.

Let λ_N denote the smallest eigenvalue of the section $\mathcal{H}_N = \{s_{j+k}\}, 0 \leq j,k \leq N$ in the positive definite case. Then (λ_N) is a decreasing sequence of positive numbers. In [1] it was proved that $\lim \lambda_N = 0$ characterizes the determinate case. The paper [2] discusses slow and rapid decrease to 0 of λ_N in the determinate case.

The orthonormal polynomials with respect to μ are denoted $(P_n), n \geq 0$.

The indeterminate case was characterized already by Hamburger by the convergence of the series $\sum |P_n(z)|^2$ for all complex z, and it leads to a reproducing kernel Hilbert space $\mathcal{E}$ of entire functions with the reproducing kernel

$$K(z,w) = \sum_{n=0}^{\infty} P_n(z)P_n(w) = \sum_{j,k=0}^{\infty} a_{j,k} z^j w^k, \quad z,w \in \mathbb{C}.$$

The coefficients $\mathcal{A} = \{a_{j,k}\}$ form an infinite symmetric matrix which is of trace class.

In work in progress [3] with Ryszard Szwarc (Wrocław) we discuss, if $\mathcal{A}$ can be considered as an inverse matrix to $\mathcal{H}$ in the sense that

$$\mathcal{H}\mathcal{A} = \mathcal{A}\mathcal{H} = \mathcal{I},$$

and the series involved in the product:

$$\sum_{k=0}^{\infty} s_{j+k} a_{k,l}$$

are absolutely convergent for all $j,l \geq 0$.

It holds for some indeterminate moment problems but not for all. The talk will give a survey of these and related results.

References

[1] C. Berg, Y. Chen, M. E. H. Ismail, *Small eigenvalues of large Hankel matrices: the indeterminate case,* Math. Scand. **91** (2002), 67–81.

[2] C. Berg and R. Szwarc, *The smallest eigenvalue of Hankel matrices*, Constr. Approx. **34** (2011), 107–133.

[3] C. Berg and R. Szwarc, *Inverse of infinite Hankel matrices.* In preparation.

Roman Bessonov (St.Petersburg)

Patrick Gerard (Paris-Orsay)

Schmidt pairs and characteristic inner functions for Hankel operators

To every singular value of a Hankel operator on the circle, I will associate an inner function allowing to describe the space of Schmidt pairs for this singular value. As an application, I will classify Hankel operators having a modulus with a finite spectrum. This talk is based on a jointwork with A. Pushnitski.

Igor Krasovsky (Imperial College London)

Nikolai Nikolskii (Bordeaux)

A glimpse into Hankel spectrum limits

We analyse some old and recent results on the asymptotics of Hankel determinants, condition numbers, and other characteristics.

Jonathan Partington (Leeds)

Numerical ranges of restricted shifts, and norms of truncated Toeplitz operators

We discuss numerical ranges of restricted shift operators and their unitary dilations, and we give an approach to calculating numerical radii via the norms of truncated Toeplitz operators (TTO) and Hankel operators.

Further results on the norm of a TTO are derived, and a conjecture on the existence of continuous symbols for compact TTO is resolved.

This is joint work with Pamela Gorkin (Bucknell) and others.

Eugene Shargorodsky (King's College London)

Pavel Šťovíček (Prague)

On the Hilbert matrix and its generalizations

The history of the Hilbert matrix is briefly reviewed while considering the Hilbert matrix as a Hankel-type operator in $\ell^2(\mathbb{Z}_+)$. Further we introduce various generalizations of the Hilbert matrix, again as matrix operators in $\ell^2(\mathbb{Z}_+)$. To this end, we need more flexibility and so the newly introduced operators are in general weighted Hankel matrices, with entries having the structure

$$B_{j,k} = w(j)w(k)h(j+k),\ \ j,k \in \mathbb{Z}_+.$$

All the discussed examples admit an explicit diagonalization. The diagonalization procedure is based on a symmetry property when a Jacobi (tridiagonal) real symmetric matrix T commuting with B is found. Every such Jacobi matrix is associated with a sequence of orthogonal polynomials and the unitary mapping diagonalizing B is described in terms of these polynomials. Thus the theory of orthogonal polynomials is an indispensable tool in the construction. Among others, in an example, based on the paper T. Kalvoda, P. Šťovíček: Linear Multilinear Alg. **64** (2016), we discuss a three-parameter family $B = B(a,b,c)$ of weighted Hankel matrices comprising the Hilbert matrix as a particular case. The corresponding orthogonal polynomials are the continuous dual Hahn polynomials.

Sergei Treil (Brown)

Dmitri Yafaev (Rennes)

On semibounded Toeplitz and Hankel operators

Necessary and sufficient conditions for Toeplitz and Hankel operators to be bounded are given by the classical theorems of Toeplitz and Nehari, respectively. Our goal is to make first steps in a study of unbounded operators of these classes. We use the Friedrichs construction of defining self-adjoint semibounded operators via corresponding quadratic forms. In the semibounded case, this construction yields most general conditions for formal symmetric operators to be defined as self-adjoint operators, but it works only if these quadratic forms are closable. So, the problem is to find necessary and sufficient conditions for Toeplitz and Hankel quadratic forms to be closable. Such conditions are found in the talk.