Spectral theory of Hankel operators and related topics
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ABSTRACTS (version 26/10/2017)

Christian Berg (Copenhagen)

Hankel matrices of indeterminate moment problems

Given a sequence $(s_n)_{n\geq 0}$ of real numbers, we consider the infinite Hankel matrix $\mathcal{H} = \{s_{j+k}\}_{j,k=0}^{\infty}$. Hamburger proved around 1920 that $\mathcal{H}$ is positive definite if and only if

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0 \tag{1}$$

for a positive measure μ on $\mathbb{R}$ with infinite support.

The moment sequence (s_n) can be **determinate** or **indeterminate** in the sense that (1) can have exactly one or several solutions μ.

Let λ_N denote the smallest eigenvalue of the section $\mathcal{H}_N = \{s_{j+k}\}, 0 \leq j,k \leq N$ in the positive definite case. Then (λ_N) is a decreasing sequence of positive numbers. In [1] it was proved that $\lim \lambda_N = 0$ characterizes the determinate case. The paper [2] discusses slow and rapid decrease to 0 of λ_N in the determinate case.

The orthonormal polynomials with respect to μ are denoted $(P_n), n \geq 0$.

The indeterminate case was characterized already by Hamburger by the convergence of the series $\sum |P_n(z)|^2$ for all complex z, and it leads to a reproducing kernel Hilbert space $\mathcal{E}$ of entire functions with the reproducing kernel

$$K(z,w) = \sum_{n=0}^{\infty} P_n(z)P_n(w) = \sum_{j,k=0}^{\infty} a_{j,k} z^j w^k, \quad z,w \in \mathbb{C}.$$

The coefficients $\mathcal{A} = \{a_{j,k}\}$ form an infinite symmetric matrix which is of trace class.

In work in progress [3] with Ryszard Szwarc (Wrocław) we discuss, if $\mathcal{A}$ can be considered as an inverse matrix to $\mathcal{H}$ in the sense that

$$\mathcal{H}\mathcal{A} = \mathcal{A}\mathcal{H} = \mathcal{I},$$

and the series involved in the product:

$$\sum_{k=0}^{\infty} s_{j+k} a_{k,l}$$

are absolutely convergent for all $j, l \geq 0$.

It holds for some indeterminate moment problems but not for all. The talk will give a survey of these and related results.

References

[1] C. Berg, Y. Chen, M. E. H. Ismail, *Small eigenvalues of large Hankel matrices: the indeterminate case,* Math. Scand. **91** (2002), 67–81.

[2] C. Berg and R. Szwarc, *The smallest eigenvalue of Hankel matrices*, Constr. Approx. **34** (2011), 107–133.

[3] C. Berg and R. Szwarc, *Inverse of infinite Hankel matrices.* In preparation.

Roman Bessonov (Chebyshev Lab, St.Petersburg)

A spectral Szegő theorem on the real line

One version of the classical Szegő theorem describes probability measures on the unit circle with finite entropy integral in terms of their recurrence (or Verblunski) coefficients. I will discuss a version of this result for even measures supported on the real line. According to Krein - de Branges inverse spectral theory, with every nonzero measure on the real line that sums up Poisson kernels one can associate the unique canonical Hamiltonian system on the positive half-axis whose Hamiltonian has unit trace almost everywhere on the half-axis. The main subject of the talk is to give a characterization of the Hamiltonians arising from even measures on the real line with finite entropy integral. No background in canonical Hamiltonian systems is assumed. This is a joint work with Serguei Denissov (University of Wisconsin-Madison).

Patrick Gérard (Paris-Orsay)

Schmidt pairs and characteristic inner functions for Hankel operators

To every singular value of a Hankel operator on the circle, I will associate an inner function allowing to describe the space of Schmidt pairs for this singular value. As an application, I will classify Hankel operators having a modulus with a finite spectrum. This talk is based on a joint work with A. Pushnitski.

Igor Krasovsky (Imperial College London)

Asymptotics of Hankel determinants

We will review results on asymptotic expansion of Hankel determinants whose symbols possess Fisher-Hartwig singularities. Some applications in the theory of random matrices will be mentioned.

Nikolai Nikolski (Bordeaux and Chebyshev Lab, St.Petersburg)

Moore-Penrose Hankel condition numbers

We estimate Moore-Penrose inverses of Hankel operators with nontrivial kernels making use techniques from the H^∞ algebra of bounded holomorphic functions. A series of integral operators on $L^2(0,1)$ is treated as an example.

Jonathan Partington (Leeds)

Numerical ranges of restricted shifts, and norms of truncated Toeplitz operators

We discuss numerical ranges of restricted shift operators and their unitary dilations, and we give an approach to calculating numerical radii via the norms of truncated Toeplitz operators (TTO) and Hankel operators.

Further results on the norm of a TTO are derived, and a conjecture on the existence of continuous symbols for compact TTO is resolved.

This is joint work with Pamela Gorkin (Bucknell) and others.

Eugene Shargorodsky (King's College London)

On the essential norms of Toeplitz operators

It is well known that the essential norm of a Toeplitz operator on the Hardy space $H^p(\mathbb{T})$, $1 < p < \infty$ is greater than or equal to the supremum norm of its symbol. In 1988, A. Böttcher, N. Krupnik, and B. Silbermann posed a question on whether or not the equality holds in the case of continuous symbols. We answer this question in the negative. On the other hand, we show that the essential norm of a Toeplitz operator with a continuous symbol is less than or equal to twice the supremum norm of the symbol and prove more precise p-dependent estimates. We also discuss some open questions related to the above estimates.

Pavel Šťovíček (Prague)

On the Hilbert matrix and its generalizations

The history of the Hilbert matrix is briefly reviewed while considering the Hilbert matrix as a Hankel-type operator in $\ell^2(\mathbb{Z}_+)$. Further we introduce various generalizations of the Hilbert matrix, again as matrix operators in $\ell^2(\mathbb{Z}_+)$. To this end, we need more flexibility and so the newly introduced operators are in general weighted Hankel matrices, with entries having the structure

$$B_{j,k} = w(j)w(k)h(j+k),\ \ j,k \in \mathbb{Z}_+.$$

All the discussed examples admit an explicit diagonalization. The diagonalization procedure is based on a symmetry property when a Jacobi (tridiagonal) real symmetric matrix T commuting with B is found. Every such Jacobi matrix is associated with a sequence of orthogonal polynomials and the unitary mapping diagonalizing B is described in terms of these polynomials. Thus the theory of orthogonal polynomials is an indispensable tool in the construction. Among others, in an example, based on the paper T. Kalvoda, P. Šťovíček: Linear Multilinear Alg. **64** (2016), we discuss a three-parameter family $B = B(a, b, c)$ of weighted Hankel matrices comprising the Hilbert matrix as a particular case. The corresponding orthogonal polynomials are the continuous dual Hahn polynomials.

Sergei Treil (Brown)

Finite rank perturbations, Clark model, and matrix weights

For a unitary operator U all its contractive perturbations $U+K$, $\|U+K\| \le 1$ by finite rank operators K with a fixed range R, $\operatorname{Ran} K \subset R$ can be parametrized by

$$T_\Gamma := U + B(\Gamma - \mathbf{I}_{\mathbb{C}^d})B^*U, \qquad \Gamma : \mathbb{C}^d \to \mathbb{C}^d,\ \|\Gamma\| \le 1,$$

where B is a fixed unitary operator $B : \mathbb{C}^d \to R$.

Under the natural assumptions that R is a *-cyclic subspace for U and Γ is a strict contraction, the operator T_Γ is the so-called *completely non-unitary* (c.n.u) contraction, so it is unitarily equivalent to its Sz.-Nagy–Foiaş functional model.

The Clark operator is a unitary operator that intertwines the operator T_Γ (which we assume is given to us in the spectral representation of the operator U) and its functional model. Description of such operator is the subject of Clark theory.

I will completely describe the Clark operator and its adjoint for the general finite rank perturbations. The adjoint Clark operator is given by the vector-valued Cauchy transform, and the direct Clark operator is given by simple algebraic formulas involving boundary values of functions from the model space. Weighted estimates with matrix valued weights appear naturally in this context.

The case of rank one perturbation of a unitary operator with purely singular spectrum was completely described (from a different point of view) by D. Clark, and later further developed by A. Aleksandrov and then by A. Poltoratski; the case of general rank one perturbations was resolved by Liaw–Treil. In the case of perturbations of rank $d > 1$, some new phenomena requiring careful investigation appear.

The talk is based on a joint work with C. Liaw.

Dmitri Yafaev (Rennes)

On semibounded Toeplitz and Hankel operators

Necessary and sufficient conditions for Toeplitz and Hankel operators to be bounded are given by the classical theorems of Toeplitz and Nehari, respectively. Our goal is to make first steps in a study of unbounded operators of these classes. We use the Friedrichs construction of defining self-adjoint semibounded operators via corresponding quadratic forms. In the semibounded case, this construction yields most general conditions for formal symmetric operators to be defined as self-adjoint operators, but it works only if these quadratic forms are closable. So, the problem is to find necessary and sufficient conditions for Toeplitz and Hankel quadratic forms to be closable. Such conditions are found in the talk.