UNIQUENESS SETS FOR THE KLEIN GORDON EQUATION AND HYPERBOLIC FOURIER SERIES

ABSTRACT. A pair (Γ, Λ), where $\Gamma \subset \mathbb{R}^2$ is a locally rectifiable curve and $\Lambda \subset \mathbb{R}^2$ is a *Heisenberg uniqueness pair* if an absolutely continuous finite complex-valued Borel measure supported on Γ whose Fourier transform vanishes on Λ necessarily is the zero measure. Hedenmalm and Montes have shown that if Γ is the hyperbola $x_1x_2 = M^2/(4\pi^2)$, $M > 0$, and Λ is the lattice-cross $(\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, where $\alpha, \beta > 0$, then (Γ, Λ) is a Heisenberg uniqueness pair if and only if $\alpha\beta M^2 \leq 4\pi^2$. The Fourier transform of a measure supported on a hyperbola solves the one-dimensional Klein-Gordon equation. By rescaling, we may assume $M = 2\pi$, and then the previous theorem is equivalent to the fact that *the functions*

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \qquad m, n \in \mathbb{Z}, \tag{1}$$

span a weak-star dense subspace of $L^\infty(\mathbb{R})$ if and only if $0 < \alpha\beta \leq 1$. The proof is based on a famous Perrón-Frobenius operator already studied by Gauss.

As for the semi-axis, that can be also stated in terms of Heisenberg uniqueness pairs, *we can show that the restriction to* $\mathbb{R}_+$ *of the functions*

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \qquad m, n \in \mathbb{Z},$$

span a weak-star dense subspace of $L^\infty(\mathbb{R}_+)$ *if and only if* $0 < \alpha\beta < 4$. In the critical regime $\alpha\beta = 4$, the weak-star span misses the mark by one dimension only. The proof of this statement is based on the ergodic properties of the standard Gauss map $t \mapsto 1/t \bmod \mathbb{Z}$ on the interval $[0, 1]$. In particular, we find that for $1 < \alpha\beta < 4$, there exist nontrivial functions $f \in L^1(\mathbb{R})$ with

$$\int_{\mathbb{R}} e^{i\pi\alpha m t} f(t)\, dt = \int_{\mathbb{R}} e^{-i\pi\beta n/t} f(t)\, dt = 0, \qquad m, n \in \mathbb{Z},$$

and that each such function is uniquely determined by its restriction to any of the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$.

As for the holomorphic counterpart of the first result, Hedenmalm and Montes, *show that the functions*

$$e^{i\pi\alpha m t}, \quad e^{-i\pi\beta n/t}, \qquad m, n \in \mathbb{Z}_+ \cup \{0\},$$

span a weak-star dense subspace of $H^\infty_+(\mathbb{R})$ *if and only if* $0 < \alpha\beta \leq 1$. Here, $H^\infty_+(\mathbb{R})$ is the subspace of $L^\infty(\mathbb{R})$ which consists of those functions whose Poisson extensions to the upper half-plane are holomorphic. In the critical regime $\alpha\beta = 1$, the proof relies on the nonexistence of a certain invariant distribution in the predual of real H^∞ for the above-mentioned Gauss-type map on the interval $[-1, 1]$. To prove it, we need to handle in a subtle way series of powers of transfer operators, a rather intractable problem. More specifically, our approach involves a splitting of the Hilbert kernel, as induced by the transfer operator. The careful analysis of this splitting involves detors to the Hurwitz zeta function as well as to the theory of totally positive matrices.

Finally, in the critical regime, $\alpha\beta = 1$, if we delete points of Λ we arrive to the new concept of Hyperbolic Fourier series. In which we find a system in $L^1(\mathbb{R})$ which is biorthogonal to (1). This is a work with Bakan, Hedenmalm, Radchenko and Viazovska.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX