Extreme eigenvalues of an integral operator

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We study the family of compact operators $B_\alpha = VA_\alpha V$, $\alpha > 0$ in $\mathsf{L}^2(\mathbb{R}^d)$, $d \geq 1$, where A_α is the pseudo-differential operator with symbol $a^{(\alpha)}(\boldsymbol{\xi}) = a(\alpha\boldsymbol{\xi})$, and both functions a and V are real-valued and decay at infinity. We assume that a and V attain their maximal values $A_0 > 0$, $V_0 > 0$ only at $\boldsymbol{\xi} = 0$ and $\mathbf{x} = 0$. We also assume that

$$a(\boldsymbol{\xi}) = A_0 - \Psi_\gamma(\boldsymbol{\xi}) + o(|\boldsymbol{\xi}|^\gamma),\ |\boldsymbol{\xi}| \to 0,$$
$$V(\mathbf{x}) = V_0 - \Phi_\beta(\mathbf{x}) + o(|\mathbf{x}|^\beta),\ |\mathbf{x}| \to 0,$$

with some functions $\Psi_\gamma(\boldsymbol{\xi}) > 0$, $\boldsymbol{\xi} \neq 0$ and $\Phi_\beta(\mathbf{x}) > 0$, $\mathbf{x} \neq 0$ that are homogeneous of degree $\gamma > 0$ and $\beta > 0$ respectively. The main result is the following asymptotic formula for the eigenvalues $\lambda_\alpha^{(n)}$ of the operator B_α (arranged in descending order counting multiplicity) for fixed n and $\alpha \to 0$:

$$\lambda_\alpha^{(n)} = A_0 V_0^2 - \mu^{(n)}\alpha^\sigma + o(\alpha^\sigma), \alpha \to 0,$$

where $\sigma^{-1} = \gamma^{-1} + \beta^{-1}$, and $\mu^{(n)}$ are the eigenvalues (arranged in ascending order counting multiplicity) of the model operator T with symbol $V_0^2\Psi_\gamma(\boldsymbol{\xi}) + 2A_0V_0\Phi_\beta(\mathbf{x})$.